$y'' = f(y')$ の形の微分方程式

例 30　次の微分方程式の一般解を求めよ。

(1)　$5\,\dfrac{d^2y}{dx^2} = \sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)\right\}^3}$　　(2)　$\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 - 1 = 0$

(解) (1)　$5\,\dfrac{d^2y}{dx^2} = \sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)\right\}^3}$

$\dfrac{dy}{dx} = p$　とおくと　$\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$　であるから

$5\,\dfrac{dp}{dx} = \left(1+p^2\right)^{\frac{3}{2}}$　$\therefore \dfrac{dx}{dp} = \dfrac{5}{(1+p^2)^{\frac{3}{2}}}$

$x = \displaystyle\int \frac{5}{(1+p^2)^{\frac{3}{2}}}dp$

ここで　$p = \tan\theta$　とおくと

$1+p^2 = 1+\tan^2\theta = \sec^2\theta$

$\dfrac{dp}{d\theta} = \sec^2\theta$　　$dp = \sec^2\theta\,d\theta$　であるから

$x = 5\displaystyle\int \frac{1}{(\sec^2\theta)^{\frac{3}{2}}}\sec^2\theta\,d\theta$

$= 5\displaystyle\int \frac{1}{\sec\theta}\,d\theta = 5\int \cos\theta\,d\theta = 5\sin\theta + C_1$

$\therefore 5\sin\theta = x - C_1$　$\therefore 25\sin^2\theta = (x-C_1)^2$

ここで　$\sin^2\theta = \tan^2\theta\cos^2\theta = \dfrac{p^2}{1+p^2}$　より

$25\,\dfrac{p^2}{1+p^2} = (x-C_1)^2$

$\therefore \{25-(x-C_1)^2\}p^2 = (x-C_1)^2$

$p^2 = \dfrac{(x-C_1)^2}{25-(x-C_1)^2}$

$p = \pm\dfrac{x-C_1}{\sqrt{25-(x-C_1)^2}}$ $\therefore \dfrac{dy}{dx} = \pm\dfrac{x-C_1}{\sqrt{5^2-(x-C_1)^2}}$

$y = \pm\displaystyle\int \frac{x-C_1}{\sqrt{5^2-(x-C_1)^2}}dx$

ここで　$x - C_1 = 5\sin\alpha$　とおくと

$\dfrac{dx}{d\alpha} = 5\cos\alpha$　　$dx = 5\cos\alpha\,d\alpha$

$\sqrt{5^2-(x-C_1)^2} = \sqrt{5^2-5^2\sin^2\alpha}$

$= \sqrt{5^2\cos^2\alpha} = 5\cos\alpha$

$\therefore y = \pm\displaystyle\int \frac{5\sin\alpha}{5\cos\alpha}5\cos\alpha\,d\alpha$

$= \pm\displaystyle\int 5\sin\alpha\,d\alpha = \mp 5\cos\alpha + C_2$

$y - C_2 = \mp 5\cos\alpha$

$(y-C_2)^2 = 25\cos^2\alpha = 25(1-\sin^2\alpha)$

$= 25-(5\sin\alpha)^2 = 25-(x-C_1)^2$

よって、求める一般解は

$(x-C_1)^2 + (y-C_2)^2 = 25$　〃

(C_1, C_2 は任意定数)

(2)　$\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 - 1 = 0$

$\dfrac{dy}{dx} = p$　とおくと　$\dfrac{dp}{dx} - p^2 - 1 = 0$

$\dfrac{dx}{dp} = \dfrac{1}{p^2+1}$　$\therefore x = \displaystyle\int \frac{1}{p^2+1}dp$

ここで　$p = \tan\theta$　とおくと

$p^2+1 = \tan^2\theta+1 = \sec^2\theta$　,　$\dfrac{dp}{d\theta} = \sec^2\theta$

$\therefore x = \displaystyle\int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta = \int d\,\theta = \theta + c$

$\therefore \theta = x - c$　　$-c = C_1$　とおくと

$\mathrm{Tan}^{-1}p = x + C_1$　　$\therefore p = \tan(x+C_1)$

$\dfrac{dy}{dx} = \tan(x+C_1)$

$y = \displaystyle\int \tan(x+C_1)dx = -\int \frac{-\sin(x+C_1)}{\cos(x+C_1)}dx$

$= -\log|\cos(x+C_1)| + C_2$

$= \log\left|\dfrac{1}{\cos(x+C_1)}\right| + C_2$

よって、求める一般解は

$y = \log|\sec(x+C_1)| + C_2$　〃

(C_1, C_2 は任意定数)