第 1I 章　微分積分 II《 § 3 重積分 》

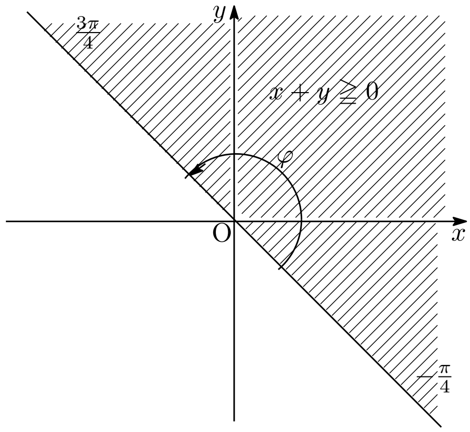
115(3)　次の広義積分の値を求めよ.

$$\iiint_V \frac{dxdydz}{(x^2+y^2+z^2+1)^2} \qquad (V\ :\ x+y \geqq 0)$$

(筑波大)

[解]　領域 $v_r\ :\ x+y \geqq 0,\ x^2+y^2+z^2 \leqq R^2\ (R>0)$　とおき,

極座標変換を用いる.



$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$x+y \geqq 0$

$V_n\ :\ 0 \leqq r \leqq R,\ 0 \leqq \theta \leqq \pi,\ -\dfrac{\pi}{4} \leqq \varphi \leqq \dfrac{3\pi}{4}$

《　ポイント　》

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\cos^2\theta\cos^2\varphi$$

$$+r^2\sin^3\theta\sin^2\varphi$$

$$+r^2\sin^3\theta\cos^2\varphi$$

$$+r^2\sin\theta\cos^2\theta\sin^2\varphi\cos\varphi$$

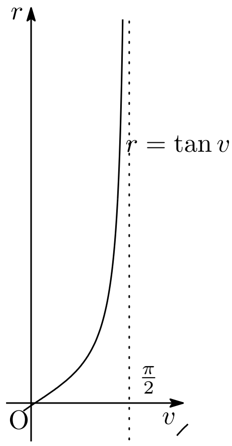
$$= r^2\sin\theta\{\sin^2\theta(\sin^2\varphi+\cos\varphi)+\cos^2\theta(\cos^2\varphi+\sin^2\varphi)\}$$

$$= r^2\sin\theta \qquad (dxdydz = r^2\sin\theta drd\theta d\varphi)$$

$$与式 = \lim_{n\to\infty}\iiint_{V_n}\frac{dxdydz}{(x^2+y^2+z^2+1)^2}$$

$$= \lim_{n\to\infty}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\pi}\int_0^R \frac{1}{(r^2+1)^2}\cdot r^2\sin\theta drd\theta d\varphi$$

$$= \lim_{n\to\infty}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\pi}\int_0^R \frac{r^2}{(r^2+1)^2}\ dr\cdot\sin\theta d\theta d\varphi$$



$r = \tan v$ とおくと,

$\dfrac{dr}{dv} = \dfrac{1}{\cos^2 v}$ 　　$\therefore\ dr = \dfrac{1}{\cos^2 v} dv$

$r^2 + 1 = \tan^2 v + 1 = \dfrac{1}{\cos^2 v}$ 　　$\therefore\ \dfrac{1}{(r^2+1)^2} = \cos^4 v$

$$r^2 = \frac{1}{\cos^2 v} - 1 = \frac{1-\cos^2 v}{\cos^2 v} = \frac{\sin^2 v}{\cos^2 v}$$

$$\int_0^R \frac{r^2}{(r^2+1)^2} dr = \int_0^{\frac{\pi}{2}} \cos^4 v \cdot \frac{\sin^2 v}{\cos^2 v} \cdot \frac{1}{\cos^2 v} dv$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 v dv = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_0^\pi \sin\theta = \Big[-\cos\theta\Big]_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) - (-1) = 2$$

$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi = \Big[\varphi\Big]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \pi$ より,

$$与式 = \lim_{n\to\infty} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^\pi \int_0^R \frac{r^2}{(r^2+1)^2}\ dr \cdot \sin\theta d\theta d\varphi$$

$$= \lim_{n\to\infty} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^\pi \frac{\pi}{4} \cdot \sin\theta d\theta d\varphi$$

$$= \lim_{n\to\infty} \frac{\pi}{4} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^\pi \sin\theta d\theta d\varphi$$

$$= \lim_{n\to\infty} \frac{\pi}{4} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2 d\varphi$$

$$= \lim_{n\to\infty} \frac{\pi}{4} \cdot 2 \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi$$

$$= \lim_{n\to\infty} \frac{\pi}{4} \cdot 2 \cdot \pi = \frac{\pi^2}{2}$$