第 4 章　応用数学《 § 1　ベクトル解析 》

237　$\boldsymbol{r} = (x,\ y,\ z),\ r = |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2}$ とする．以下の量を計算せよ．

(1) $\boldsymbol{r}$ の勾配 $\nabla \boldsymbol{r}$

(2) $\dfrac{1}{\boldsymbol{r}}$ の勾配 $\nabla \dfrac{1}{\boldsymbol{r}}$

(3) $\boldsymbol{r}$ の発散 $\nabla \cdot \boldsymbol{r}$

(4) $\boldsymbol{w} = (0,\ 0,\ w)$ (w は正の定数) とするときの，$\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$ の回転 $\nabla \times \boldsymbol{v}$

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《 ポイント 》φ をスカラー場，$\boldsymbol{a} = (a_1,\ a_2,\ a_3)$ をベクトル場とする．

勾配 $grad\,\varphi = \nabla\varphi = \left(\dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}\right)\ \varphi = \left(\dfrac{\partial\varphi}{\partial x},\ \dfrac{\partial\varphi}{\partial y},\ \dfrac{\partial\varphi}{\partial z}\right)$

点 p における，単位ベクトル $\boldsymbol{e}$ 方向の方向微分係数 $(\nabla\varphi)p \cdot \boldsymbol{e}$

発散 $div\,\boldsymbol{a} = \nabla \cdot \boldsymbol{a} = \left(\dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}\right) \cdot (a_1,\ a_2,\ a_3) = \dfrac{\partial a_1}{\partial x} + \dfrac{\partial a_2}{\partial y} + \dfrac{\partial a_3}{\partial z}$

回転; $rot\,\boldsymbol{a} = \nabla \times \boldsymbol{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \left(\dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z},\ \dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x},\ \dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y}\right)$

[解]

(1) $r = |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$ より

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x) = x(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2y) = y(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2z) = z(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

$$\therefore\ \nabla r = \left(\frac{\partial r}{\partial x},\ \frac{\partial r}{\partial y},\ \frac{\partial r}{\partial z}\right) = \left(\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right) = \frac{1}{r}(x,\ y,\ z) = \frac{1}{r}\boldsymbol{r} = \frac{\boldsymbol{r}}{r}$$

(2) 《 ポイント 》$\nabla\, f(r) = f'(r)\,\nabla\, r$

$$\nabla\,\frac{1}{r} = \nabla\, r^{-1} = (r^{-1})'\,\nabla\, r = (-1)r^{-2}\,\nabla\, r = -\frac{1}{r^2}\,\nabla\, r = -\frac{1}{r^2}\,\frac{\boldsymbol{r}}{r} = -\frac{\boldsymbol{r}}{r^3}$$

[別解]　$r^{-1} = \left\{(x^2+y^2+z^2)^{\frac{1}{2}}\right\}^{-1} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\frac{\partial r^{-1}}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}(2x)$$

$$= -x(x^2+y^2+z^2)^{-\frac{3}{2}} = -x\left\{(x^2+y^2+z^2)^{\frac{1}{2}}\right\}^{-3} = -x\,r^{-3} = -\frac{x}{r^3}$$

同様にして　$\dfrac{\partial r^{-1}}{\partial y} = -\dfrac{y}{r^3}$，$\dfrac{\partial r^{-1}}{\partial z} = -\dfrac{z}{r^3}$

$$\therefore\ \nabla\,\frac{1}{r} = \nabla r^{-1} = \left(\frac{\partial r^{-1}}{\partial x},\ \frac{\partial r^{-1}}{\partial y},\ \frac{\partial r^{-1}}{\partial z}\right) = \left(-\frac{x}{r^3},\ -\frac{y}{r^3},\ -\frac{z}{r^3}\right) = -\frac{1}{r^3}(x,\ y,\ z) = -\frac{1}{r^3}\boldsymbol{r} = -\frac{\boldsymbol{r}}{r^3}$$

(3) 《　ポイント　》$\boldsymbol{r}$ の発散 $V \cdot \boldsymbol{r}$

$$V \cdot \boldsymbol{r} = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) \cdot (x,\ y,\ z) = \frac{x}{x} + \frac{y}{y} + \frac{z}{z} = 1 + 1 + 1 = 3$$

(4) $\boldsymbol{r} = (x,\ y,\ z),\ r = \sqrt{x^2 + y^2 + z^2},\ \boldsymbol{w} = (0,\ 0,\ w)$　(w は正の定数) より

$$\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = (0 - wy,\ wx - 0,\ 0 - 0) = (-wy,\ wx,\ 0)$$

$$V \times \boldsymbol{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = \left(\frac{\partial\, 0}{\partial y} - \frac{\partial\, wx}{\partial z},\ \frac{\partial\, (-wy)}{\partial z} - \frac{\partial\, 0}{\partial x},\ \frac{\partial\, wx}{\partial x} - \frac{\partial\, (-wy)}{\partial y}\right)$$

$$= \left(0 - \frac{\partial\, wx}{\partial z},\ \frac{\partial\, (-wy)}{\partial z} - 0,\ \frac{\partial\, wx}{\partial x} - \frac{\partial\, (-wy)}{\partial y}\right)$$

$$= \left(-\frac{\partial\, wx}{\partial z},\ \frac{\partial\, (-wy)}{\partial z},\ w - (-w)\right)$$

$$= (0,\ 0,\ 2w) = 2(0,\ 0,\ w) = 2\boldsymbol{w}$$