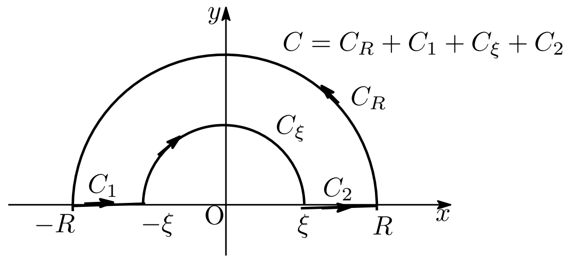
第 4 章　応用数学《 § 3 複素関数 》

$\boxed{271}$ $\displaystyle\int_C \frac{e^{iz}}{z}dz$ を右図のような

複素平面上の経路 C で計算することにより,

$\displaystyle\int_0^{\infty} \frac{\sin x}{x}dx$ の値を求めよ.



(お茶の水大)

《　ポイント　》$\displaystyle\int_{C_R} \frac{e^{iz}}{z}dz \longrightarrow 0$ を示すために, $\sin t \geqq \dfrac{2}{\pi}t \ \left(0 \leqq t \leqq \dfrac{\pi}{2}\right)$ を使う.

[解] $f(z) = \dfrac{e^{iz}}{z}$ とおく. $f(z)$ の特異点は $z = 0$ で 1 位の極である.

積分経路を $C_R : z = Re^{it}$ $(0 \leqq t \leqq \pi)$　　$C_1 : z = t$ $(-R \leqq t \leqq -\xi)$　　$C_2 : z = t$ $(\xi \leqq t \leqq \pi)$

C_ξ の逆向き $-C_\xi = \xi e^{it}$ $(0 \leqq t \leqq \pi)$　として,

$C = C_1 + C_\xi + C_2 + C_R$　とおく.

$f(z)$ は曲線 C の内部と周上で正則だから, $\displaystyle\int_C f(z)dz = 0 \cdots ①$

《　ポイント　》$t = -s$ とおくと, $\dfrac{dt}{ds} = -1 \ \therefore\ dt = (-1)ds$

$$\int_{C_1} f(z)dz = \int_{-R}^{-\xi} f(t)dt = \int_R^{\xi} f(-s)\cdot(-1)ds = \int_{\xi}^{R} f(-s)ds$$

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = \int_{\xi}^{R} f(-t)dt + \int_{\xi}^{R} f(t)dt$$

$$= \int_{\xi}^{R} (f(-t) + f(t))\,dt = \int_{\xi}^{R} \left(\frac{e^{i(-t)}}{-t} + \frac{e^{it}}{t}\right)dt = \int_{\xi}^{R} \frac{e^{it} - e^{-it}}{t}dt$$

$$= \int_{\xi}^{R} \frac{2i}{t}\cdot\frac{e^{it} - e^{-it}}{2i}dt$$

《　ポイント　》$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$,　$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \int_{\xi}^{R} \frac{2i}{t}\cdot\sin t dt = 2i\int_{\xi}^{R} \frac{\sin t}{t}dt; \cdots ②$$

$$\int_{\xi} f(z)dz = \int_{\pi}^{0} \frac{e^{i\xi e^{it}}}{\xi e^{it}}\cdot i\xi e^{it}dt = -i\int_0^{\pi} e^{i\xi e^{it}}dt \ \text{だから,}$$

$$\lim_{\xi \to +0} \int_{C_\xi} f(t)dt = -i\int_0^{\pi} e^0 dt = -i\int_0^{\pi} dt = -i\Big[\ t\ \Big]_0^{\pi} = -i\pi \ \cdots ③$$

$$\left|\int_{C_R} f(z)dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}\cdot iRe^{it}dt\right|$$

$$\leqq \int_0^{\pi} \left|ie^{iRe^{it}}\right| dt$$

《　ポイント　》$2i\sin\theta = e^{i\theta} - e^{-i\theta}$　　$2\cos\theta = e^{i\theta} + e^{-i\theta}$

$2i\sin\theta + 2\cos\theta = 2e^{it}$　　$\therefore\ i\sin\theta + \cos\theta = e^{it}$

$$= \int_0^{\pi} \left|ie^{iR(i\sin t + \cos t)}\right| dt$$

$$= \int_0^{\pi} \left|ie^{R(i^2\sin t + i\cos t)}\right| dt$$

$$= \int_0^{\pi} \left|ie^{iR\cos t}e^{R(-\sin t)}\right| dt$$

$$= \int_0^{\pi} \left|ie^{iR\cos t}e^{-R\sin t}\right| dt = \int_0^{\pi} e^{-R\sin t}dt$$

$$= \int_0^{\frac{\pi}{2}} e^{-R\sin t}dt + \int_{\frac{\pi}{2}}^{\pi} e^{-R\sin t}dt$$

与式 $= \displaystyle\int_0^{\frac{\pi}{2}} e^{-R\sin t}dt + \int_{\frac{\pi}{2}}^0 e^{-R\sin t}dt$

　ここで，$t = \pi - r$ おくと，$\dfrac{dt}{dr} = -1$　　$\therefore\ dt = -dr$

$\displaystyle\int_{\frac{\pi}{2}}^0 e^{-R\sin(\pi - r)}(-1)dr = \int_0^{\frac{\pi}{2}} e^{-R\sin r}dr$ だから，

与式$= \displaystyle\int_0^{\frac{\pi}{2}} e^{-R\sin t}dt + \int_0^{\frac{\pi}{2}} e^{-R\sin t}dt$

$= 2\displaystyle\int_0^{\frac{\pi}{2}} e^{-R\sin t}dt$　　　　《　ポイント　》$\sin t = \dfrac{2}{\pi}t$　$\left(0 \leqq t \leqq \dfrac{\pi}{2}\right)$ を使う．

$\leqq 2\displaystyle\int_0^{\frac{\pi}{2}} e^{-R\cdot\frac{2}{\pi}t}dt$

$= 2\left[-\dfrac{\pi}{2R}e^{\frac{2R}{\pi}t}\right]_0^{\frac{\pi}{2}} = -\dfrac{\pi}{R}\left[e^{\frac{2R}{\pi}t}\right]_0^{\frac{\pi}{2}}$

$= -\dfrac{\pi}{R}\left(e^{\frac{2R}{\pi}\cdot\frac{\pi}{2}} - e^0\right) = -\dfrac{\pi}{R}\left(e^R - 1\right) = \dfrac{\pi}{R}(1 - e^R)$

これより，

$\displaystyle\lim_{R\to\infty}\left|\int_{C_R} f(z)dz\right| \leqq \lim_{R\to\infty}\frac{\pi}{R}\left(1 - e^R\right) = 0$

したがって，$\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)dz = 0\ \cdots④$

$\displaystyle\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{C_1} f(z)dz + \int_\xi f(z)dz + \int_{C_2} f(z)dz$

ここで，①，② より，$\displaystyle\int_C f(z)dz = 0,$　　$\displaystyle\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 2i\int_\xi^R \frac{\sin x}{x}dx$ だから，

$0 = 2i\displaystyle\int_\xi^R \frac{\sin x}{x}dx + \int_{C_R} f(z)dz + \int_\xi f(z)dz$

$\xi \to +0$，　$R \to \infty$ とすると，

③，④ より，$\displaystyle\lim_{\xi\to+0}\int_\xi f(z)dz = -i\pi,$　　$\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ だから，

$0 = 2i\displaystyle\int_\xi^R \frac{\sin x}{x}dx + 0 - i\pi$

よって，$\displaystyle\int_\xi^R \frac{\sin x}{x}dx = \frac{\pi}{2}$