積分法 ≪基礎≫ 小テスト　$(No.10)$　解答例

1. 次の定積分の値を求めよ。

(1) $\displaystyle\int_0^3 \frac{1}{(x^2+9)^2}dx$　　≪考え方≫ $\displaystyle\int_0^3 \frac{1}{(x^2+9)^2}dx = \int_0^3 \frac{1}{(x^2+3^2)^2}dx$ と変形してみる。

（解）$x = 3\tan\theta\ (0 \leqq \theta \leqq \frac{\pi}{4})$ とおくと、

$\displaystyle\frac{dx}{d\theta} = 3\sec^2\theta \quad \therefore dx = 3\sec^2\theta d\theta$

x	$0 \to 3$
θ	$0 \to \frac{\pi}{4}$

≪計算≫
$x=0$ のとき
$0 = 3\tan\theta$
$\therefore \tan\theta = 0 \quad \therefore \theta = 0$
$x=3$ のとき
$3 = 3\tan\theta$
$\therefore \tan\theta = 1 \quad \therefore \theta = \frac{\pi}{4}$

$(x^2+9)^2 = (9\tan^2\theta+9)^2 = \{9(\tan^2\theta+1)\}^2 = (9\sec^2\theta)^2 = 81\sec^4\theta$

$$\int_0^3 \frac{1}{(x^2+9)^2}dx = \int_0^{\frac{\pi}{4}} \frac{1}{81\sec^4\theta}\cdot 3\sec^2\theta d\theta = \frac{1}{27}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}d\theta$$

$$= \frac{1}{27}\int_0^{\frac{\pi}{4}} \cos^2\theta d\theta = \frac{1}{27}\int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2}d\theta = \frac{1}{54}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)d\theta$$

$$= \frac{1}{54}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{54}\left\{\left(\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0+\frac{1}{2}\sin 0\right)\right\} = \frac{1}{54}\left(\frac{\pi}{4}+\frac{1}{2}\times 1\right) = \frac{1}{216}(\pi+2) \quad〃$$

(2) $\displaystyle\int_1^2 \sqrt{4-x^2}dx$　　≪考え方≫ $\displaystyle\int_1^2 \sqrt{4-x^2}dx = \int_1^2 \sqrt{2^2-x^2}dx$ と変形してみる。

（解 1）$x = 2\sin\theta\ (\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{2})$ とおくと、

$\displaystyle\frac{dx}{d\theta} = 2\cos x \quad \therefore dx = 2\cos x d\theta$

x	$1 \to 2$
θ	$\frac{\pi}{6} \to \frac{\pi}{2}$

≪計算≫
$x=1$ のとき
$1 = 2\sin\theta$
$\therefore \sin\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{6}$
$x=2$ のとき
$2 = 2\sin\theta$
$\therefore \sin\theta = 1 \quad \therefore \theta = \frac{\pi}{2}$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta}$

$= \sqrt{(2\cos\theta)^2} = |2\cos\theta| = 2\cos\theta$

$$\int_1^2 \sqrt{4-x^2}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos\theta\cdot 2\cos\theta d\theta = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}d\theta = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta = 2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left\{\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right) - \left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right\} = 2\left\{\left(\frac{\pi}{2}+\frac{1}{2}\cdot 0\right) - \left(\frac{\pi}{6}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right\}$$

$$= 2\left(\frac{\pi}{2}-\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right) = 2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) = \frac{2}{3}\pi - \frac{\sqrt{3}}{2} \quad〃$$

（解 2）公式 $\displaystyle\int\sqrt{a^2-x^2}dx = \frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\ \mathrm{Sin}^{-1}\frac{x}{a}\right)$ を利用すると

$$\int_1^2 \sqrt{4-x^2}dx = \int_1^2 \sqrt{2^2-x^2}dx = \left[\frac{1}{2}\left(x\sqrt{2^2-x^2}+2^2\ \mathrm{Sin}^{-1}\frac{x}{2}\right)\right]_1^2$$

$$= \frac{1}{2}\left(2\sqrt{0}+4\ \mathrm{Sin}^{-1}1\right) - \frac{1}{2}\left(1\cdot\sqrt{3}+4\ \mathrm{Sin}^{-1}\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(0+4\cdot\frac{\pi}{2}\right) - \frac{1}{2}\left(\sqrt{3}+4\cdot\frac{\pi}{6}\right) = \pi - \frac{\sqrt{3}}{2} - \frac{\pi}{3} = \frac{2}{3}\pi - \frac{\sqrt{3}}{2} \quad〃$$

（解 3）積分の値は右図の砂地の面積 S に等しいから

扇形 $OAB = \frac{1}{2}\cdot 2^2\cdot\frac{\pi}{3} = \frac{2}{3}\pi$　，　$\triangle OBC = \frac{1}{2}\cdot 1\cdot\sqrt{3} = \frac{\sqrt{3}}{2}$

$$\int_1^2 \sqrt{4-x^2}dx = S = 扇形\ OAB - \triangle OBC = \frac{2}{3}\pi - \frac{\sqrt{3}}{2} \quad〃$$