積分法 ≪基礎≫ 小テスト　$(No.13)$　解答例

1. 次の定積分の値を求めよ。

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}dx$$

（解）与式 $= \displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{2^2-x^2}}dx = \left[\ \mathrm{Sin}^{-1}\frac{x}{2}\right]_{\sqrt{2}}^{\sqrt{3}} = \mathrm{Sin}^{-1}\frac{\sqrt{3}}{2} - \mathrm{Sin}^{-1}\frac{\sqrt{2}}{2}$

$= \mathrm{Sin}^{-1}\dfrac{\sqrt{3}}{2} - \mathrm{Sin}^{-1}\dfrac{1}{\sqrt{2}}$

$= \dfrac{\pi}{3} - \dfrac{\pi}{4} = \dfrac{4}{12}\pi - \dfrac{3}{12}\pi = \dfrac{\pi}{12}$　〃

（別解）与式$= \displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{2^2-x^2}}dx$

＜ 計算 ＞

$x=\sqrt{2}$のとき	$x=\sqrt{3}$のとき
$\sqrt{2}=2\sin\theta$	$\sqrt{3}=2\sin\theta$
$\sin\theta=\frac{1}{\sqrt{2}}$	$\sin\theta=\frac{\sqrt{3}}{2}$
$\theta=\frac{\pi}{4}$	$\theta=\frac{\pi}{3}$

$x = 2\sin\theta \left(\dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{3}\right)$ とおくと

$\dfrac{dx}{d\theta} = 2\cos\theta$　$\therefore dx = 2\cos\theta d\theta$

x	$\sqrt{2}$	$\to$	$\sqrt{3}$
θ	$\frac{\pi}{4}$	$\to$	$\frac{\pi}{3}$

$\sqrt{2^2-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = \sqrt{(2\cos\theta)^2} = |2\cos\theta| = 2\cos\theta$

与式$= \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2\cos\theta}\cdot 2\cos\theta d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1\, d\theta = \left[\ \theta\ \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{4}{12}\pi - \frac{3}{12}\pi = \frac{\pi}{12}$　〃

2. 次の定積分の値を求めよ。

$$\int_{-1}^{1} \sqrt{4-x^2}dx$$

（解）与式 $= 2\displaystyle\int_0^1 \sqrt{2^2-x^2}dx = 2\cdot\frac{1}{2}\Big[x\sqrt{2^2-x^2} + 2^2\ \mathrm{Sin}^{-1}\frac{x}{2}\Big]_0^1$

$= \Big[x\sqrt{4-x^2} + 4\ \mathrm{Sin}^{-1}\dfrac{x}{2}\Big]_0^1 = \left(1\cdot\sqrt{3} + 4\cdot\ \mathrm{Sin}^{-1}\dfrac{1}{2}\right) - \left(0\cdot\sqrt{4} + 4\cdot\ \mathrm{Sin}^{-1}0\right)$

$= \sqrt{3} + 4\cdot\dfrac{\pi}{6} = \sqrt{3} + \dfrac{2}{3}\pi$　〃

（別解）与式$=2\displaystyle\int_0^1 \sqrt{2^2-x^2}dx$

＜ 計算 ＞

$x=0$ のとき	$x=1$ のとき
$0=2\sin\theta$	$1=2\sin\theta$
$\sin\theta=0$	$\sin\theta=\frac{1}{2}$
$\theta=0$	$\theta=\frac{\pi}{6}$

$x = 2\sin\theta \left(0 \leqq \theta \leqq \dfrac{\pi}{6}\right)$ とおくと

$\dfrac{dx}{d\theta} = 2\cos\theta$　$\therefore dx = 2\cos\theta d\theta$

x	0	$\to$	1
θ	0	$\to$	$\frac{\pi}{6}$

$\sqrt{2^2-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = \sqrt{(2\cos\theta)^2} = |2\cos\theta| = 2\cos\theta$

与式$=2\displaystyle\int_0^{\frac{\pi}{6}} 2\cos\theta\cdot 2\cos\theta d\theta = 8\int_0^{\frac{\pi}{6}} \cos^2\theta d\theta = 8\int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2}d\theta$

$= 4\displaystyle\int_0^{\frac{\pi}{6}} (1+\cos 2\theta)d\theta = 4\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{6}} = 4\left\{\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - \left(0 + \frac{1}{2}\sin 0\right)\right\}$

$= 4\left(\dfrac{\pi}{6} + \dfrac{1}{2}\cdot\dfrac{\sqrt{3}}{2}\right) = \dfrac{2}{3}\pi + \sqrt{3}$　〃

3. 次の定積分の値を求めよ。

$\displaystyle\int_1^3 \frac{dx}{x^2+3}dx$

（解） 与式 $\displaystyle=\int_1^3\frac{dx}{x^2+(\sqrt{3})^2}dx=\left[\frac{1}{\sqrt{3}}\ \mathrm{Tan}^{-1}\frac{x}{\sqrt{3}}\right]_1^3=\frac{1}{\sqrt{3}}\left[\ \mathrm{Tan}^{-1}\frac{x}{\sqrt{3}}\right]_1^3$

$\displaystyle=\frac{1}{\sqrt{3}}\left(\ \mathrm{Tan}^{-1}\frac{3}{\sqrt{3}}-\mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}\left(\ \mathrm{Tan}^{-1}\sqrt{3}-\ \mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}\right)$

$\displaystyle=\frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}\left(\frac{2\pi}{6}-\frac{\pi}{6}\right)=\frac{\pi}{6\sqrt{3}}=\frac{\sqrt{3}}{18}\pi$　〃

（別解） 与式 $\displaystyle=\int_1^3\frac{1}{x^2+(\sqrt{3})^2}dx$

＜ 計算 ＞

$x=1$ のとき	$x=3$ のとき
$1=\sqrt{3}\tan\theta$	$3=\sqrt{3}\tan\theta$
$\tan\theta=\frac{1}{\sqrt{3}}$	$\tan\theta=\sqrt{3}$
$\theta=\frac{\pi}{6}$	$\theta=\frac{\pi}{3}$

$x=\sqrt{3}\tan\theta\left(\dfrac{\pi}{6}\leqq\theta\leqq\dfrac{\pi}{3}\right)$ とおくと

$\dfrac{dx}{d\theta}=\sqrt{3}\sec^2\theta$　$\therefore dx=\sqrt{3}\sec^2\theta d\theta$

x	1	$\to$	3
θ	$\frac{\pi}{6}$	$\to$	$\frac{\pi}{3}$

$x^2+(\sqrt{3})^2=3\tan^2\theta+3=3(1+\tan^2\theta)=3\sec^2\theta$

与式 $\displaystyle=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{3\sec^2\theta}\cdot\sqrt{3}\sec^2\theta d\theta=\frac{1}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}1\ d\theta=\frac{1}{\sqrt{3}}\Big[\ \theta\ \Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}=\frac{\sqrt{3}}{18}\pi$　〃

4. 次の極限値を定積分の式で表せ。また、その値を求めよ。

$\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+n}$

（解） 与式 $\displaystyle=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{1}{n}}{\frac{2k}{n}+\frac{n}{n}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{1}{n}}{2(\frac{k}{n})+1}=\int_0^1\frac{dx}{2x+1}=\int_0^1\frac{1}{2x+1}dx$

$\displaystyle=\frac{1}{2}\int_0^1\frac{2}{2x+1}dx=\frac{1}{2}\int_0^1\frac{(2x+1)'}{2x+1}dx=\frac{1}{2}\Big[\log|2x+1|\Big]_0^1=\frac{1}{2}(\log 3-\log 1)=\frac{1}{2}\log 3$　〃

5. 次の極限値を定積分の式で表せ。また、その値を求めよ。

$\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2+3n^2}$

≪考え方≫
$n\to\infty$ のとき　$\frac{k}{n}$ の値は
$\frac{1}{n}\to 0$，　$\frac{n}{n}\to 1$
$\therefore 0\leqq\frac{k}{n}\leqq 1$,
$\therefore 0\leqq x_k\leqq 1$
よって区間 [0,1] を考える。

(解) 与式 $\displaystyle=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{n}{n^2}}{\frac{k^2}{n^2}+\frac{3n^2}{n^2}}$

$\displaystyle=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{1}{n}}{\left(\frac{k}{n}\right)^2+3}=\int_0^1\frac{dx}{x^2+3}$

$\displaystyle=\int_0^1\frac{1}{x^2+(\sqrt{3})^2}dx=\left[\frac{1}{\sqrt{3}}\ \mathrm{Tan}^{-1}\frac{x}{\sqrt{3}}\right]_0^1$

$\displaystyle=\frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}0=\frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}-\frac{1}{\sqrt{3}}\cdot 0=\frac{\pi}{6\sqrt{3}}$　〃

6. $\displaystyle\int_1^x f(t)dt=x^2-ax+2$ を満たす関数 $f(x)$ と定数 a を求めよ。　**≪公式≫** $\displaystyle\frac{d}{dx}\int_1^x f(t)dt=f(x)$ **の利用**

(解) $\displaystyle\frac{d}{dx}\int_1^x f(t)dt=\frac{d}{dx}(x^2-ax+2)$　　$f(x)=2x-a$　　$\therefore f(t)=2t-a$

$\displaystyle\int_1^x f(t)dt=\int_1^x(2t-a)dt=\Big[t^2-at\Big]_1^x=(x^2-ax)-(1^2-a\cdot 1)=x^2-ax+a-1$

これと与式を比べて　　$a-1=2$　　$\therefore a=3$　〃　，　$\therefore f(x)=2x-3$　〃