積分法 ≪基礎≫ 小テスト　$(No.5)$　解答例

1. 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{(x+1)(x^2+1)}{x^2}dx$

（解） $\displaystyle\int \frac{(x+1)(x^2+1)}{x^2}dx = \int \frac{x^3+x^2+x+1}{x^2}dx = \int \left(x+1+\frac{1}{x}+\frac{1}{x^2}\right)dx$

$\displaystyle= \int \left(x+1+\frac{1}{x}+x^{-2}\right)dx = \frac{1}{1+1}x^{1+1}+1\cdot x+\log|x|+\frac{1}{-2+1}x^{-2+1}$

$\displaystyle= \frac{1}{2}x^2+x+\log|x|-x^{-1} = \frac{1}{2}x^2+x-\frac{1}{x}+\log|x|$ 〃

(2) $\displaystyle\int \left(\frac{1}{\sqrt{3+x^2}}-\frac{1}{\sqrt{3-x^2}}\right)dx$

（解） $\displaystyle\int \left(\frac{1}{\sqrt{3+x^2}}-\frac{1}{\sqrt{3-x^2}}\right)dx = \int \frac{dx}{\sqrt{3+x^2}} - \int \frac{dx}{\sqrt{(\sqrt{3})^2-x^2}}$

$\displaystyle= \log\left|x+\sqrt{3+x^2}\right| - \mathrm{Sin}^{-1}\frac{x}{\sqrt{3}}$ 〃

2. 次の定積分の値を求めよ。

(1) $\displaystyle\int_0^{\pi}(e^x-\sin x)\,dx$

（解） $\displaystyle\int_0^{\pi}(e^x-\sin x)\,dx = \Big[e^x+\cos x\Big]_0^{\pi} = (e^{\pi}+\cos\pi)-\left(e^0+\cos 0\right)$

$= (e^{\pi}-1)-(1+1) = e^{\pi}-3$ 〃

(2) $\displaystyle\int_1^3 \frac{dx}{x^2+3}$

（解） $\displaystyle\int_1^3 \frac{dx}{x^2+3} = \int_1^3 \frac{1}{x^2+(\sqrt{3})^2}dx = \left[\frac{1}{\sqrt{3}}\ \mathrm{Tan}^{-1}\frac{x}{\sqrt{3}}\right]_1^3 = \frac{1}{\sqrt{3}}\left[\ \mathrm{Tan}^{-1}\frac{x}{\sqrt{3}}\right]_1^3$

$\displaystyle= \frac{1}{\sqrt{3}}\left(\ \mathrm{Tan}^{-1}\sqrt{3}-\ \mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{6\sqrt{3}}$ 〃

3. 半角の公式を用いて、定積分 $\displaystyle\int_0^{\frac{\pi}{3}}\cos^2\frac{x}{2}\,dx$ の値を求めよ。

（解）半角の公式より、$\displaystyle\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$ であるから

$\displaystyle\int_0^{\frac{\pi}{3}}\cos^2\frac{x}{2}\,dx = \int_0^{\frac{\pi}{3}}\frac{1+\cos x}{2}dx = \frac{1}{2}\int_0^{\frac{\pi}{3}}(1+\cos x)dx = \frac{1}{2}\Big[x+\sin x\Big]_0^{\frac{\pi}{3}}$

$\displaystyle= \frac{1}{2}\left\{\left(\frac{\pi}{3}+\sin\frac{\pi}{3}\right)-(0+\sin 0)\right\} = \frac{1}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}+\frac{\sqrt{3}}{4}$ 〃

≪参考≫　2 倍角の公式 $\cos 2\alpha = \cos^2\alpha-\sin^2\alpha = 2\cos^2\alpha-1$ より、$2\cos^2\alpha = 1+\cos 2\alpha$

$\displaystyle\therefore\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$　　$\displaystyle\alpha = \frac{\theta}{2}$ とおくと、半角の公式 $\displaystyle\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$ を得る。

記憶しておかないと、対応が困難な公式

公式　$\displaystyle\int \frac{1}{\sqrt{x^2+A}}dx = \log\left|x+\sqrt{x^2+A}\,\right| \qquad (A \neq 0)$

(証明) $\displaystyle\left(\log\left|x+\sqrt{x^2+A}\,\right|\right)' = \frac{1}{x+\sqrt{x^a+A}}\cdot\left\{1+\frac{(x^2+A)'}{2\sqrt{x^2+A}}\right\}$

$$= \frac{1}{x+\sqrt{x^a+A}}\cdot\left\{1+\frac{2x}{2\sqrt{x^2+A}}\right\} = \frac{1}{x+\sqrt{x^a+A}}\cdot\frac{\sqrt{x^2+A}+x}{\sqrt{x^2+A}} = \frac{1}{\sqrt{x^a+A}}$$

$$\therefore \int \frac{1}{\sqrt{x^2+A}}dx = \log\left|x+\sqrt{x^2+A}\,\right| \quad 〃$$

忘れた場合は置換積分で対応できる公式

公式　$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \mathrm{Sin}^{-1}\frac{x}{a} \qquad (a>0)$

(証明 1) $x = a\sin\theta$ とおくと $\displaystyle\frac{dx}{d\theta} = a\cos\theta \quad \therefore dx = a\cos\theta d\theta$

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = |a\cos\theta\,| = a\cos\theta$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{a\cos\theta}\cdot a\cos\theta = \int 1\, d\theta = \theta$$

$x = a\sin\theta$ より $\displaystyle\sin\theta = \frac{x}{a} \quad \therefore \theta = \mathrm{Sin}^{-1}\frac{x}{a}$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}}dx = \mathrm{Sin}^{-1}\frac{x}{a} \quad 〃$$

(証明 2) $\displaystyle\left(\mathrm{Sin}^{-1}\frac{x}{a}\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\frac{x}{a}\right)' = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}}dx = \mathrm{Sin}^{-1}\frac{x}{a} \quad 〃$$

公式　$\displaystyle\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} \qquad (a \neq 0)$

(証明 1) $x = a\tan\theta$ とおくと $\displaystyle\frac{dx}{d\theta} = a\sec^2\theta \quad \therefore dx = a\sec^2\theta d\theta$

$$x^2+a^2 = a^2\tan^2\theta + a^2 = a^2(\tan^2\theta+1) = a^2\sec^2\theta$$

$$\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2\sec^2\theta}\cdot a\sec^2\theta\, d\theta = \frac{1}{a}\int 1\, d\theta = \frac{1}{a}\theta$$

$x = a\tan\theta$ より $\displaystyle\tan\theta = \frac{x}{a} \quad \therefore \theta = \mathrm{Tan}^{-1}\frac{x}{a}$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} \quad 〃$$

(証明 2) $\displaystyle\left(\frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a}\right)' = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\left(\frac{x}{a}\right)' = \frac{1}{a}\cdot\frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a} = \frac{1}{a^2}\cdot\frac{1}{1+\frac{x^2}{a^2}} = \frac{1}{a^2+x^2}$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} \quad 〃$$