積分法 ≪基礎≫ 小テスト　$(No.7)$　解答例

1. 次の定積分の値を求めよ。

(1) $\displaystyle\int_1^2 (3x-5)^4 dx$

（解）$t=3x-5$ とおくと、$\dfrac{dt}{dx}=3$　よって　$\dfrac{1}{3}dt=dx$

$x=1$ のとき $t=-2$,　　$x=2$ のとき $t=1$

x	1	$\to$	2
t	-2	$\to$	1

$$\int_1^2 (3x-5)^4 dx=\int_{-2}^1 t^4\cdot\frac{1}{3}dt=\frac{1}{3}\int_{-2}^1 t^4 dt=\frac{1}{3}\left[\frac{1}{5}t^5\right]_{-2}^1=\frac{1}{15}\left[t^5\right]_{-2}^1$$

$$=\frac{1}{15}\left\{1^5-(-2)^5\right\}=\frac{1}{15}\left\{1-(-32)\right\}=\frac{33}{15}=\frac{11}{5}\ 〃$$

（別解）$\displaystyle\int x^4 dx=\frac{1}{5}x^5$ であるから

$$\int_1^2 (3x-5)^4 dx=\left[\frac{1}{3}\cdot\frac{1}{5}(3x-5)^5\right]_1^2=\frac{1}{15}\left[(3x-5)^5\right]_1^2=\frac{1}{5}\left\{1^5-(-2)^5\right\}$$

$$=\frac{1}{15}\left\{1-(-32)\right\}=\frac{33}{15}=\frac{11}{5}\ 〃$$

(2) $\displaystyle\int_0^{\frac{\pi}{3}} \cos^3 x\sin x dx$

（解）$t=\cos x$ とおくと、$\dfrac{dt}{dx}=-\sin x$　$\therefore (-1)dt=\sin x dx$

$x=0$ のとき $t=\cos 0=1,\ x=\frac{\pi}{3}$ のとき $t=\cos\frac{\pi}{3}=\frac{1}{2}$

x	0	$\to$	$\frac{\pi}{3}$
t	1	$\to$	$\frac{1}{2}$

$$与式=\int_1^{\frac{1}{2}} t^3\cdot(-1)dt=-\int_1^{\frac{1}{2}} t^3 dt=-\left[\frac{1}{4}t^4\right]_1^{\frac{1}{2}}=-\frac{1}{4}\left[t^4\right]_1^{\frac{1}{2}}$$

$$=-\frac{1}{4}\left\{\left(\frac{1}{2}\right)^4-1^4\right\}=-\frac{1}{4}\left(\frac{1}{16}-1\right)=-\frac{1}{4}\left(-\frac{15}{16}\right)=\frac{15}{64}\ 〃$$

（途中から別解）

$$\int_0^{\frac{\pi}{3}}\cos^3 x\sin x dx=\int_1^{\frac{1}{2}} t^3\cdot(-1)dt=\int_{\frac{1}{2}}^1 t^3 dt=\left[\frac{1}{4}t^4\right]_{\frac{1}{2}}^1=\frac{1}{4}\left[t^4\right]_{\frac{1}{2}}^1$$

$$=\frac{1}{4}\left\{1^4-\left(\frac{1}{2}\right)^4\right\}=\frac{1}{4}\left(1-\frac{1}{16}\right)=\frac{15}{64}\ 〃$$

(3) $\displaystyle\int_e^{e^3}\frac{dx}{x\log x}$

（解）$t=\log x$ とおくと、$\dfrac{dt}{dx}=\dfrac{1}{x}$　$\therefore dt=\dfrac{1}{x}dx$

$x=e$ のとき $t=\log e=1$,　$x=e^3$のとき

$t=\log e^3=3\log e=3\times 1=3$

x	e	$\to$	e^3
t	1	$\to$	3

$$\int_e^{e^3}\frac{dx}{x\log x}=\int_e^{e^3}\frac{1}{\log x}\cdot\frac{1}{x}dx=\int_1^3\frac{1}{t}dt=\left[\log|t|\right]_1^3=\log 3-\log 1=\log 3\ 〃$$

（別解）$\displaystyle\int_e^{e^3}\frac{dx}{x\log x}=\int_e^{e^3}\frac{\frac{1}{x}}{\log x}dx=\int_e^{e^3}\frac{(\log x)'}{\log x}dx=\left[\log|\log x|\right]_e^{e^3}$

$$=\log|\log e^3|-\log|\log e|=\log|3\log e|-\log 1=\log(3\cdot 1)-0=\log 3\ 〃$$

2. 次の定積分の値を求めよ。

$\displaystyle\int_0^{\frac{3}{2}}\sqrt{9-x^2}dx$　　≪考え方≫ $\displaystyle\int_0^{\frac{3}{2}}\sqrt{3^2-x^2}dx$ と変形する。

（解）$x=3\sin\theta\ (0\leqq\theta\leqq\frac{\pi}{6})$ とおくと、

$$\frac{dx}{d\theta}=3\cos\theta\quad\therefore dx=3\cos\theta d\theta$$

$$\sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=\sqrt{9(1-\sin^2\theta)}=\sqrt{9\cos^2\theta}$$

$$=\sqrt{(3\cos\theta)^2}=|3\cos\theta|=3\cos\theta$$

$$\int_0^{\frac{3}{2}}\sqrt{9-x^2}dx=\int_0^{\frac{\pi}{6}}3\cos\theta\cdot3\cos\theta d\theta=9\int_0^{\frac{\pi}{6}}\cos^2\theta d\theta$$

$$=9\int_0^{\frac{\pi}{6}}\frac{1+\cos2\theta}{2}d\theta=\frac{9}{2}\int_0^{\frac{\pi}{6}}(1+\cos2\theta)d\theta$$

$$=\frac{9}{2}\Big[\theta+\frac{1}{2}\sin2\theta\Big]_0^{\frac{\pi}{6}}=\frac{9}{2}\left\{\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)-\left(0+\frac{1}{2}\sin0\right)\right\}$$

$$=\frac{9}{2}\left(\frac{\pi}{6}+\frac{1}{2}\times\frac{\sqrt{3}}{2}\right)=\frac{3}{4}\pi+\frac{9\sqrt{3}}{8}\quad〃$$

≪計算≫
$x=0$　のとき
$0=3\sin\theta$
$\therefore\sin\theta=0\quad\therefore\theta=0$
$x=\frac{3}{2}$　のとき
$\frac{3}{2}=3\sin\theta$
$\therefore\sin\theta=\frac{1}{2}\quad\therefore\theta=\frac{\pi}{6}$

x	0	$\to$	$\frac{3}{2}$
θ	0	$\to$	$\frac{\pi}{6}$

（別解）与式 $\displaystyle=\int_0^{\frac{3}{2}}\sqrt{3^2-x^2}dx=\frac{1}{2}\Big[x\sqrt{3^2-x^2}+3^2\ \mathrm{Sin}^{-1}\frac{x}{3}\Big]_0^{\frac{3}{2}}$

$$=\tfrac{1}{2}\Big\{\big(\tfrac{3}{2}\sqrt{9-\tfrac{9}{4}}+9\ \mathrm{Sin}^{-1}\tfrac{1}{2}\big)-\big(0\sqrt{9-0}+9\ \mathrm{Sin}^{-1}0\big)\Big\}$$

$$=\frac{1}{2}\Big(\frac{3}{2}\sqrt{\frac{27}{4}}+9\cdot\frac{\pi}{6}\Big)=\frac{1}{2}\Big(\frac{3}{2}\cdot\frac{3\sqrt{3}}{2}+\frac{3}{2}\pi\Big)=\frac{9\sqrt{3}}{8}+\frac{3}{4}\pi\quad〃$$

a が正の定数のとき，公式 $\displaystyle\int\sqrt{a^2-x^2}dx=\frac{1}{2}\Big(x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\frac{x}{a}\Big)$ を証明せよ.

(証明１)　$I=\displaystyle\int\sqrt{a^2-x^2}dx$ とおくと，$\displaystyle\int1dx=x$ であるから，部分積分法によって

$$I=\int1\cdot\sqrt{a^2-x^2}dx=x\sqrt{a^2-x^2}-\int x\left(\sqrt{a^2-x^2}\right)'dx=x\sqrt{a^2-x^2}-\int x\cdot\frac{-2x}{2\left(\sqrt{a^2-x^2}\right)}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{-x^2}{\sqrt{a^2-x^2}}dx=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\left\{\int\sqrt{a^2-x^2}dx-\int\frac{a^2}{\sqrt{a^2-x^2}}dx\right\}$$

$$=x\sqrt{a^2-x^2}-\left\{I-a^2\int\frac{1}{\sqrt{a^2-x^2}}dx\right\}=x\sqrt{a^2-x^2}-I+a^2\mathrm{Sin}^{-1}\frac{x}{a}$$

これから　$2I=x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\dfrac{x}{a}$　　$\therefore$　$I=\dfrac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\dfrac{x}{a}\right)$　〃

(証明２)　$\displaystyle\left(x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\frac{x}{a}\right)'=1\cdot\sqrt{a^2-x^2}+x\cdot\frac{(a^2-x^2)'}{2\sqrt{a^2-x^2}}+a^2\cdot\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\frac{x}{a}\right)'$

$$=\sqrt{a^2-x^2}+x\cdot\frac{-2x}{2\sqrt{a^2-x^2}}+a^2\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}=\sqrt{a^2-x^2}-\frac{x^2}{\sqrt{a^2-x^2}}+\frac{a^2}{\sqrt{a^2-x^2}}$$

$$=\sqrt{a^2-x^2}+\frac{a^2-x^2}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}+\sqrt{a^2-x^2}=2\sqrt{a^2-x^2}$$

$$\left\{\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\frac{x}{a}\right)\right\}'=\sqrt{a^2-x^2}\quad\therefore\int\sqrt{a^2-x^2}dx=\frac{1}{2}\Big(x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\frac{x}{a}\Big)\quad〃$$