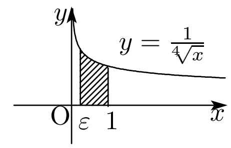
積分の応用 ≪基礎≫ 小テスト　解答例　$(No.10)$

1. 次の広義積分の値を求めよ。

(1) $\displaystyle\int_0^1 \frac{1}{\sqrt[4]{x}}dx$



（解）$y=\dfrac{1}{\sqrt[4]{x}}$ は $x=0$ を除いた区間 $(0,1]$ で連続であるから

$\displaystyle\int_0^1 \frac{1}{\sqrt[4]{x}}dx=\lim_{\varepsilon\to+0}\int_\varepsilon^1 \frac{1}{\sqrt[4]{x}}dx=\lim_{\varepsilon\to+0}\int_\varepsilon^1 \frac{1}{x^{\frac{1}{4}}}dx=\lim_{\varepsilon\to+0}\int_\varepsilon^1 x^{-\frac{1}{4}}dx$

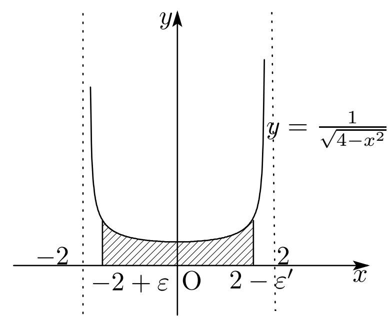
$\displaystyle=\lim_{\varepsilon\to+0}\left[\frac{1}{-\frac{1}{4}+1}x^{-\frac{1}{4}+1}\right]_\varepsilon^1=\lim_{\varepsilon\to+0}\left[\frac{4}{3}x^{\frac{3}{4}}\right]_\varepsilon^1=\lim_{\varepsilon\to+0}\frac{4}{3}\left[\sqrt[4]{x^3}\right]_\varepsilon^1$

$\displaystyle=\lim_{\varepsilon\to+0}\frac{4}{3}\left(\sqrt[4]{1^3}-\sqrt[4]{\varepsilon^3}\right)=\frac{4}{3}(1-0)=\frac{4}{3}$　〃

≪略記≫ $\displaystyle\int_0^1 \frac{1}{\sqrt[4]{x}}dx=\int_0^1 \frac{1}{x^{\frac{1}{4}}}dx=\int_0^1 x^{-\frac{1}{4}}dx=\left[\frac{1}{-\frac{1}{4}+1}x^{-\frac{1}{4}+1}\right]_0^1$

$\displaystyle=\left[\frac{4}{3}x^{\frac{3}{4}}\right]_0^1=\frac{4}{3}\left[\sqrt[4]{x^3}\right]_0^1=\frac{4}{3}(1-0)=\frac{4}{3}$　〃

(2) $\displaystyle\int_{-2}^2 \frac{1}{\sqrt{4-x^2}}dx$



（解）$y=\dfrac{1}{\sqrt{4-x^2}}$ は $x=\pm2$ を除く区間 $(-2,2)$ で
連続であるから

$\displaystyle\int_{-2}^2 \frac{1}{\sqrt{4-x^2}}dx=\lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\int_{-2+\varepsilon}^{2-\varepsilon'} \frac{1}{\sqrt{2^2-x^2}}dx$

$\displaystyle=\lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\left[\ \mathrm{Sin}^{-1}\frac{x}{2}\right]_{-2+\varepsilon}^{2-\varepsilon'}$

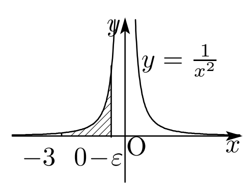
$\displaystyle=\lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\left(\ \mathrm{Sin}^{-1}\frac{2-\varepsilon'}{2}-\mathrm{Sin}^{-1}\frac{-2+\varepsilon}{2}\right)$

$\displaystyle=\mathrm{Sin}^{-1}1-\mathrm{Sin}^{-1}(-1)=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$　〃

≪略記≫ $\displaystyle\int_{-2}^2 \frac{1}{\sqrt{4-x^2}}dx=\int_{-2}^2 \frac{1}{\sqrt{2^2-x^2}}dx=\left[\ \mathrm{Sin}^{-1}\frac{x}{2}\right]_{-2}^2$

$\displaystyle=\mathrm{Sin}^{-1}1-\mathrm{Sin}^{-1}(-1)=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$　〃

(3) $\displaystyle\int_{-3}^0 \frac{1}{x^2}dx$



（解）$y=\dfrac{1}{x^2}$ は $x=0$ を除く区間で $[-3,0)$ 連続であるから

$\displaystyle\int_{-3}^0 \frac{1}{x^2}dx=\lim_{\varepsilon\to+0}\int_{-3}^{0-\varepsilon} \frac{1}{x^2}dx=\lim_{\varepsilon\to+0}\int_{-3}^{-\varepsilon} x^{-2}dx$

$\displaystyle=\lim_{\varepsilon\to+0}\left[\frac{1}{-2+1}x^{-2+1}\right]_{-3}^{-\varepsilon}=\lim_{\varepsilon\to+0}\left[-x^{-1}\right]_{-3}^{-\varepsilon}$

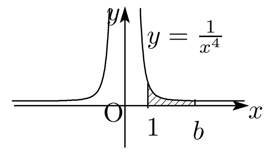
$\displaystyle=\lim_{\varepsilon\to+0}\left[-\frac{1}{x}\right]_{-3}^{-\varepsilon}=\lim_{\varepsilon\to+0}\left\{\left(-\frac{1}{-\varepsilon}\right)-\left(-\frac{1}{-3}\right)\right\}$

$\displaystyle=\lim_{\varepsilon\to+0}\left(\frac{1}{\varepsilon}-\frac{1}{3}\right)=+\infty$

よって、値は存在しない。

2. 次の広義積分の値を求めよ。

(1) $\displaystyle\int_1^{\infty}\frac{1}{x^4}dx$



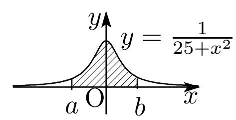
（解）$y=\dfrac{1}{x^4}$ は区間 $[1,\infty)$ で連続であるから

$$\int_1^{\infty}\frac{1}{x^4}dx=\lim_{b\to\infty}\int_1^{b}\frac{1}{x^4}dx=\lim_{b\to\infty}\int_1^{b}x^{-4}dx=\lim_{b\to\infty}\left[\frac{1}{-4+1}x^{-4+1}\right]_1^b=\lim_{b\to\infty}\left[-\frac{1}{3}x^{-3}\right]_1^b$$

$$=\lim_{b\to\infty}\left[-\frac{1}{3}\cdot\frac{1}{x^3}\right]_1^b=\lim_{b\to\infty}\left[-\frac{1}{3x^3}\right]_1^b=\lim_{b\to\infty}\left\{\left(-\frac{1}{-3b^3}\right)-\left(-\frac{1}{3}\right)\right\}=\frac{1}{3}\quad〃$$

(2) $\displaystyle\int_{-\infty}^{\infty}\frac{1}{25+x^2}dx$

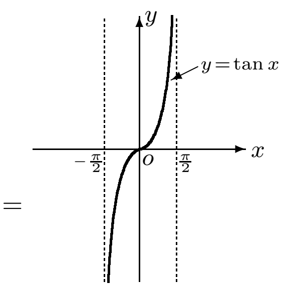
（解）$y=\dfrac{1}{25+x^2}$ は区間 $(-\infty,\infty)$ で連続であるから



$$\int_{-\infty}^{\infty}\frac{1}{25+x^2}dx=\lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b\frac{1}{5^2+x^2}dx$$

$$=\lim_{\substack{a\to-\infty\\ b\to\infty}}\left[\frac{1}{5}\tan^{-1}\frac{x}{5}\right]_a^b=\lim_{\substack{a\to-\infty\\ b\to\infty}}\frac{1}{5}\left[\tan^{-1}\frac{x}{5}\right]_a^b$$

$$=\lim_{\substack{a\to-\infty\\ b\to\infty}}\frac{1}{5}\left(\mathrm{Tan}^{-1}\frac{b}{5}-\mathrm{Tan}^{-1}\frac{a}{5}\right)=\frac{1}{5}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}=\frac{\pi}{5}\quad〃$$

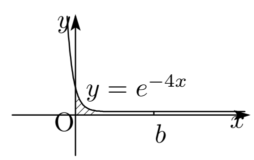


≪略記≫ $\displaystyle\int_{-\infty}^{\infty}\frac{1}{25+x^2}dx=\int_{-\infty}^{\infty}\frac{1}{5^2+x^2}dx=\left[\frac{1}{5}\tan^{-1}\frac{x}{5}\right]_{-\infty}^{\infty}=$

$\displaystyle\frac{1}{5}\left[\tan^{-1}\frac{x}{5}\right]_{-\infty}^{\infty}=\frac{1}{5}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}=\frac{\pi}{5}\quad〃$

3. 次の広義積分を求めよ。

(1) $\displaystyle\int_0^{\infty}e^{-4x}dx$



（解）$y=e^{-4x}$ は区間 $[0,\infty)$ で連続であるから

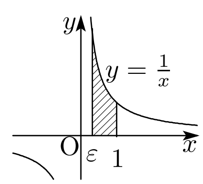
$$\int_0^{\infty}e^{-4x}dx=\lim_{b\to\infty}\int_0^b e^{-4x}dx=\lim_{b\to\infty}\left[\frac{1}{-4}e^{-4x}\right]_0^b$$

$$=\lim_{b\to\infty}\left[-\frac{1}{4}\cdot\frac{1}{e^{4x}}\right]_0^b=\lim_{b\to\infty}\left[-\frac{1}{4e^{4x}}\right]_0^b$$

$$=\lim_{b\to\infty}\left\{\left(-\frac{1}{4e^{4b}}\right)-\left(-\frac{1}{4e^{0}}\right)\right\}$$

$$=\lim_{b\to\infty}\left(-\frac{1}{4e^{4b}}+\frac{1}{4}\right)=0+\frac{1}{4}=\frac{1}{4}\quad〃$$

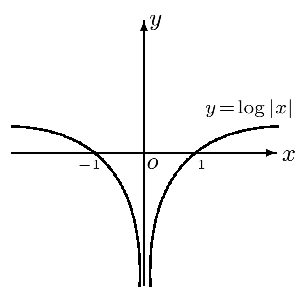
(2) $\displaystyle\int_0^{1}\frac{1}{x}dx$

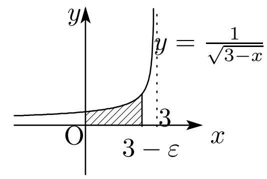


（解）$y=\dfrac{1}{x}$ は $x=0$ を除いた区間 $(0,1]$ で連続であるから

$$\int_0^1\frac{1}{x}dx=\lim_{\varepsilon\to+0}\int_{\varepsilon}^1\frac{1}{x}dx=\lim_{\varepsilon\to+0}\Big[\log|x|\Big]_{\varepsilon}^1=\lim_{\varepsilon\to+0}(\log 1-\log\varepsilon)$$

$=\displaystyle\lim_{\varepsilon\to+0}(-\log\varepsilon)=-(-\infty)=+\infty$　　よって、値は存在しない。





(3) $\displaystyle\int_0^3 \frac{1}{\sqrt{3-x}}dx$

（解１）$y=\dfrac{1}{\sqrt{3-x}}$ は $x=3$ を除いた区間 $[\,0,3\,)$ で連続であるから

$$\int_0^3 \frac{1}{\sqrt{3-x}}dx=\lim_{\varepsilon\to+0}\int_0^{3-\varepsilon}\frac{1}{\sqrt{3-x}}dx$$

$t=3-x$ とおくと　$\dfrac{dt}{dx}=-1$　$\therefore dt=(-1)dx$　$\therefore (-1)dt=dx$

x	$0\ \to\ 3-\varepsilon$
t	$3\ \to\ \varepsilon$

$$\therefore 与式=\lim_{\varepsilon\to+0}\int_3^{\varepsilon}\frac{1}{\sqrt{t}}\cdot(-1)dt=\lim_{\varepsilon\to+0}\int_{\varepsilon}^3\frac{1}{\sqrt{t}}dt=\lim_{\varepsilon\to+0}\int_{\varepsilon}^3 t^{-\frac{1}{2}}dt=\lim_{\varepsilon\to+0}\Big[\frac{1}{-\frac{1}{2}+1}t^{-\frac{1}{2}+1}\Big]_{\varepsilon}^3$$

$$=\lim_{\varepsilon\to+0}\Big[2t^{\frac{1}{2}}\Big]_{\varepsilon}^3=\lim_{\varepsilon\to+0}\Big[2\sqrt{t}\Big]_{\varepsilon}^3=\lim_{\varepsilon\to+0}(2\sqrt{3}-2\sqrt{\varepsilon})=2\sqrt{3}\quad〃$$

≪略記１≫ $\displaystyle\int_0^3\frac{1}{\sqrt{3-x}}dx$　　$t=3-x$ とおくと　$\dfrac{dt}{dx}=-1$　$\therefore dt=(-1)dx$　$\therefore (-1)dt=dx$

x	$0\ \to\ 3$
t	$3\ \to\ 0$

$$\therefore 与式=\int_3^0\frac{1}{\sqrt{t}}\cdot(-1)dt=\int_0^3\frac{1}{\sqrt{t}}dt=\int_0^3 t^{-\frac{1}{2}}dt=\Big[\frac{1}{-\frac{1}{2}+1}t^{-\frac{1}{2}+1}\Big]_0^3=\Big[2t^{\frac{1}{2}}\Big]_0^3=\Big[2\sqrt{t}\Big]_0^3=(2\sqrt{3}-2\sqrt{0})=2\sqrt{3}\quad〃$$

（解２）$y=\dfrac{1}{\sqrt{3-x}}$ は $x=3$ を除いた区間 $[\,0,3\,)$ で連続であるから

$$\int_0^3\frac{1}{\sqrt{3-x}}dx=\lim_{\varepsilon\to+0}\int_0^{3-\varepsilon}\frac{1}{\sqrt{3-x}}dx=\lim_{\varepsilon\to+0}\int_0^{3-\varepsilon}(3-x)^{-\frac{1}{2}}dx=\lim_{\varepsilon\to+0}\Big[\frac{1}{-1}\cdot\frac{1}{-\frac{1}{2}+1}(3-x)^{-\frac{1}{2}+1}\Big]_0^{3-\varepsilon}$$

$$=\lim_{\varepsilon\to+0}\Big[-2(3-x)^{\frac{1}{2}}\Big]_0^{3-\varepsilon}=\lim_{\varepsilon\to+0}\Big[-2\sqrt{3-x}\Big]_0^{3-\varepsilon}=\lim_{\varepsilon\to+0}\Big\{\big(-2\sqrt{3-(3-\varepsilon)}\big)-\big(-2\sqrt{3-0}\big)\Big\}$$

$$=\lim_{\varepsilon\to+0}\big(-2\sqrt{\varepsilon}+2\sqrt{3}\big)=2\sqrt{3}\quad〃$$

≪略記２≫
$$\int_0^3\frac{1}{\sqrt{3-x}}dx=\int_0^3(3-x)^{-\frac{1}{2}}dx=\Big[\frac{1}{-1}\cdot\frac{1}{-\frac{1}{2}+1}(3-x)^{-\frac{1}{2}+1}\Big]_0^3=\Big[-2(3-x)^{\frac{1}{2}}\Big]_0^3$$

$$=\Big[-2\sqrt{3-x}\Big]_0^3=(-2\sqrt{0})-(-2\sqrt{3})=2\sqrt{3}\quad〃$$

（解3）
$$\int\frac{1}{\sqrt{3-x}}dx=\int\frac{1}{(3-x)^{\frac{1}{2}}}dx=\int(3-x)^{-\frac{1}{2}}dx=\frac{1}{-1}\cdot\frac{1}{-\frac{1}{2}+1}(3-x)^{-\frac{1}{2}+1}=-2(3-x)^{\frac{1}{2}}=-2\sqrt{3-x}$$

$y=\dfrac{1}{\sqrt{3-x}}$ は $x=3$ を除いた区間 $[\,0,3\,)$ で連続であるから

$$\int_0^3\frac{1}{\sqrt{3-x}}dx=\lim_{\varepsilon\to+0}\int_0^{3-\varepsilon}\frac{1}{\sqrt{3-x}}dx=\lim_{\varepsilon\to+0}\Big[-2\sqrt{3-x}\Big]_0^{3-\varepsilon}$$

$$=\lim_{\varepsilon\to+0}\Big\{\big(-2\sqrt{3-(3-\varepsilon)}\big)-\big(-2\sqrt{3-0}\big)\Big\}$$

$$=\lim_{\varepsilon\to+0}\big(-2\sqrt{\varepsilon}+2\sqrt{3}\big)=2\sqrt{3}\quad〃$$

≪略記３≫
$$\int\frac{1}{\sqrt{3-x}}dx=\int\frac{1}{(3-x)^{\frac{1}{2}}}dx=\int(3-x)^{-\frac{1}{2}}dx$$

$$=\frac{1}{-1}\cdot\frac{1}{-\frac{1}{2}+1}(3-x)^{-\frac{1}{2}+1}=-2(3-x)^{\frac{1}{2}}=-2\sqrt{3-x}$$

$$\int_0^3\frac{1}{\sqrt{3-x}}dx=\Big[-2\sqrt{3-x}\Big]_0^3=(-2\sqrt{3-3})-(-2\sqrt{3-0})=2\sqrt{3}\quad〃$$