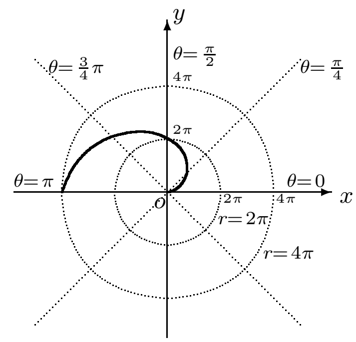
積分の応用 ≪基礎≫ 小テスト　解答例　($No.8$)

1. 曲線 C : $r=4\theta\ (0 \leqq \theta \leqq \pi)$ と半直線 $\theta=\pi$ で囲まれる図形を A とする。このとき、次の各問いに答えよ。



(1) 次の表の空白を埋め、曲線 C の概形を描け。

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π
r	0	π	2π	3π	4π

(2) 図形 A の面積を求めよ。

(解) $S=\dfrac{1}{2}\displaystyle\int_0^{\pi} r^2 d\theta=\dfrac{1}{2}\int_0^{\pi}(4\theta)^2 d\theta=\dfrac{1}{2}\int_0^{\pi}16\theta^2 d\theta=8\int_0^{\pi}\theta^2 d\theta$

$=8\Big[\dfrac{1}{3}\theta^3\Big]_0^{\pi}=\dfrac{8}{3}\Big[\theta^3\Big]_0^{\pi}=\dfrac{8}{3}(\pi^3-0)=\dfrac{8}{3}\pi^3$　〃

(3) 曲線 C の長さを求めよ。

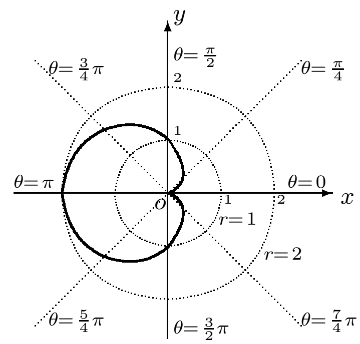
(解) $\dfrac{dr}{d\theta}=4$　　$\therefore \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}=\sqrt{(4\theta)^2+4^2}=\sqrt{16\theta^2+16}=\sqrt{16(\theta^2+1)}=4\sqrt{\theta^2+1}$

$$L=\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=\int_0^{\pi}4\sqrt{\theta^2+1}\,d\theta=4\int_0^{\pi}\sqrt{\theta^2+1}\,d\theta$$

$$=4\cdot\frac{1}{2}\Big[\theta\sqrt{\theta^2+1}+1\cdot\log\left|\theta+\sqrt{\theta^2+1}\right|\Big]_0^{\pi}=2\Big[\theta\sqrt{\theta^2+1}+\log\left|\theta+\sqrt{\theta^2+1}\right|\Big]_0^{\pi}$$

$$=2\left\{\left(\pi\sqrt{\pi^2+1}+\log\left|\pi\sqrt{\pi^2+1}\right|\right)-(0+\log 1)\right\}=2\left\{\pi\sqrt{\pi^2+1}+\log\left(\pi+\sqrt{\pi^2+1}\right)\right\}$$　〃

2. 曲線 C : $r=1-\cos\theta\ (0 \leqq \theta \leqq 2\pi)$ で囲まれる図形を A とする。このとき、次の各問いに答えよ。



(1) 次の表の空白を埋め、曲線 C の概形を描け。

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
r	0	$1-\frac{\sqrt{2}}{2}$	1	$1+\frac{\sqrt{2}}{2}$	2	$1+\frac{\sqrt{2}}{2}$	1	$1-\frac{\sqrt{2}}{2}$	0

(2) 図形 A の面積を求めよ。

(解) $S=2\cdot\dfrac{1}{2}\displaystyle\int_0^{\pi}r^2 d\theta=\int_0^{\pi}r^2 d\theta=\int_0^{\pi}(1-\cos\theta)^2 d\theta$

$=\displaystyle\int_0^{\pi}\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta=4\int_0^{\pi}\sin^4\frac{\theta}{2}d\theta$

半角の公式
$\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}$
$\therefore 1-\cos\theta=2\sin^2\frac{\theta}{2}$

$t=\dfrac{\theta}{2}$　とおくと　$\dfrac{dt}{d\theta}=\dfrac{1}{2}$　$\therefore 2dt=d\theta$

θ	$0 \to \pi$
t	$0 \to \frac{\pi}{2}$

$S=4\displaystyle\int_0^{\frac{\pi}{2}}\sin^4 t\cdot 2dt=8\int_0^{\frac{\pi}{2}}\sin^4 t dt=8\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{2}\pi$　〃

(3) 曲線 C の長さを求めよ。

(解) $\dfrac{dr}{d\theta}=0-(-\sin\theta)=\sin\theta$

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{(1-\cos\theta)^2+(\sin\theta)^2}=\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}=\sqrt{2(1-\cos\theta)}$$

$$=\sqrt{2\cdot 2\sin^2\frac{\theta}{2}}=\sqrt{\left(2\sin\frac{\theta}{2}\right)^2}=\left|2\sin\frac{\theta}{2}\right|=2\sin\frac{\theta}{2}$$　$\left(\because 0\leqq\dfrac{\theta}{2}\leqq\pi \text{より}\quad \sin\dfrac{\theta}{2}\geqq 0\right)$

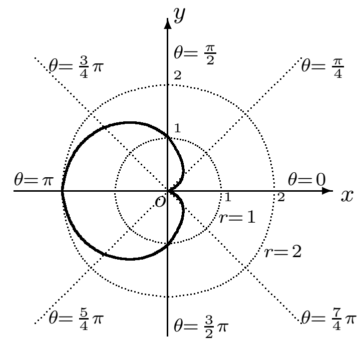
$$L=2\cdot\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=2\int_0^{\pi}2\sin\frac{\theta}{2}d\theta=4\int_0^{\pi}\sin\frac{\theta}{2}d\theta$$

$t=\dfrac{\theta}{2}$　とおくと　$\dfrac{dt}{d\theta}=\dfrac{1}{2}$　$\therefore 2dt=d\theta$

θ	$0 \to \pi$
t	$0 \to \frac{\pi}{2}$

$L=4\displaystyle\int_0^{\frac{\pi}{2}}\sin t\cdot 2dt=8\int_0^{\frac{\pi}{2}}\sin t dt=8\Big[-\cos t\Big]_0^{\frac{\pi}{2}}=-8\Big[\cos t\Big]_0^{\frac{\pi}{2}}=-8\left(\cos\frac{\pi}{2}-\cos 0\right)=8$　〃

2． 曲線 C : $r = 1 - \cos\theta\ (0 \leqq \theta \leqq 2\pi)$ で囲まれる図形を A とする。このとき、次の各問いに答えよ。



(1) 次の表の空白を埋め、曲線 C の概形を描け。

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
r	0	$1-\frac{\sqrt{2}}{2}$	1	$1+\frac{\sqrt{2}}{2}$	2	$1+\frac{\sqrt{2}}{2}$	1	$1-\frac{\sqrt{2}}{2}$	0
		(0.3)		(1.7)		(1.7)		(0.3)	

(2) 図形 A の面積を求めよ。

(別解 1) $S = 2\cdot\dfrac{1}{2}\displaystyle\int_0^{\pi} r^2 d\theta = \int_0^{\pi} r^2 d\theta = \int_0^{\pi} (1-\cos\theta)^2 d\theta$

$$= \int_0^{\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta = \int_0^{\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= \left(\frac{3}{2}\pi - 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - \left(0 - 2\sin 0 + \frac{1}{4}\sin 0\right) = \frac{3}{2}\pi \quad〃$$

(別解 2) $S = \dfrac{1}{2}\displaystyle\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 d\theta$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left\{\left(3\pi - 2\sin 2\pi + \frac{1}{4}\sin 4\pi\right) - \left(0 - 2\sin 0 + \frac{1}{4}\sin 0\right)\right\} = \frac{3}{2}\pi \quad〃$$

(3) 曲線 C の長さを求めよ。

(別解 1) $\dfrac{dr}{d\theta} = 0 - (-\sin\theta) = \sin\theta$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2(1-\cos\theta)}$$

$$= \sqrt{2\cdot 2\sin^2\frac{\theta}{2}} = \sqrt{\left(2\sin\frac{\theta}{2}\right)^2} = \left|2\sin\frac{\theta}{2}\right| = 2\sin\frac{\theta}{2} \qquad \left(\because 0 \leqq \frac{\theta}{2} \leqq \pi \text{より}\quad \sin\frac{\theta}{2} \geqq 0\right)$$

$$L = 2\cdot\int_0^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2\int_0^{\pi} 2\sin\frac{\theta}{2}d\theta = 4\int_0^{\pi}\sin\frac{\theta}{2}d\theta$$

$$t = 4\left[\frac{1}{\frac{1}{2}}\cdot\left(-\cos\frac{\theta}{2}\right)\right]_0^{\pi} = \left[2\left(-\cos\frac{\theta}{2}\right)\right]_0^{\pi} = -8\left[\cos\frac{\theta}{2}\right]_0^{\pi} = -8\left(\cos\frac{\pi}{2} - \cos 0\right) = 8 \quad〃$$

(別解 2) $\dfrac{dr}{d\theta} = 0 - (-\sin\theta) = \sin\theta$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2(1-\cos\theta)}$$

$$= \sqrt{2\cdot 2\sin^2\frac{\theta}{2}} = \sqrt{\left(2\sin\frac{\theta}{2}\right)^2} = \left|2\sin\frac{\theta}{2}\right| = 2\sin\frac{\theta}{2} \quad \left(\because 0 \leqq \frac{\theta}{2} \leqq \pi \text{より } \sin\frac{\theta}{2} \geqq 0\right)$$

$$L = \int_0^{2\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}d\theta = 2\int_0^{2\pi}\sin\frac{\theta}{2}d\theta$$

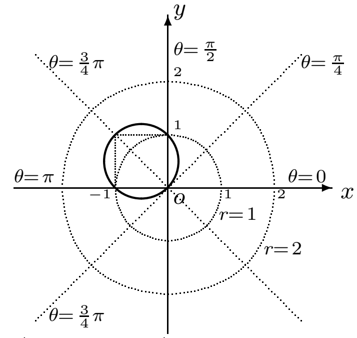
$$t = 2\left[\frac{1}{\frac{1}{2}}\cdot\left(-\cos\frac{\theta}{2}\right)\right]_0^{2\pi} = 2\left[2\left(-\cos\frac{\theta}{2}\right)\right]_0^{2\pi} = -4\left[\cos\frac{\theta}{2}\right]_0^{2\pi} = -4\left(\cos\pi - \cos 0\right) = 8 \quad〃$$

《類題の研究》

1. 曲線 $C : r = \sin\theta - \cos\theta$ $\left(\frac{\pi}{4} \leqq \theta \leqq \frac{5}{4}\pi\right)$ に囲まれる図形を A とする。このとき、次の各問いに答えよ。

(1) 次の表の空白を埋め、曲線 C の概形を描け。

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$
r	0	1	$\sqrt{2}$	1	0



(2) 図形 A の面積を求めよ。

(解) $S = 2 \cdot \dfrac{1}{2}\displaystyle\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} r^2 d\theta = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (\sin\theta - \cos\theta)^2 d\theta$

$$= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)d\theta = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (1 - \sin 2\theta)d\theta$$

$$= \left[\theta + \frac{1}{2}\cos 2\theta\right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} = \left(\frac{3}{4}\pi + \frac{1}{2}\cos\frac{3}{2}\pi\right) - \left(\frac{\pi}{4} + \frac{1}{2}\cos\frac{\pi}{2}\right) = \frac{3}{4}\pi + \frac{1}{2}\cdot 0 - \frac{\pi}{4} - \frac{1}{2}\cdot 0 = \frac{\pi}{2}$$ 〃

(3) 曲線 C の長さを求めよ。

(解) $\dfrac{dr}{d\theta} = \cos\theta - (-\sin\theta) = \cos\theta + \sin\theta$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(\sin\theta - \cos\theta)^2 + (\cos\theta + \sin\theta)^2}$$

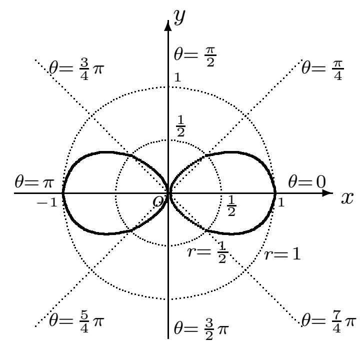
$$= \sqrt{\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta + \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta} = \sqrt{2(\sin^2\theta + \cos^2\theta)} = \sqrt{2}$$

$$L = \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} \sqrt{2}\, d\theta = \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} 1 d\theta = \sqrt{2}\Big[\theta\Big]_{\frac{\pi}{4}}^{\frac{5}{4}\pi} = \sqrt{2}\left(\frac{5}{4}\pi - \frac{\pi}{4}\right) = \sqrt{2}\pi$$ 〃

2. 曲線 C : $r = \cos^2\theta$ $(0 \leqq \theta \leqq 2\pi)$ で囲まれる図形を A とする。このとき、次の各問いに答えよ。

(1) 次の表の空白を埋め、曲線 C の概形を描け。

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
r	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1



(2) 図形 A の面積を求めよ。

(解) $S = 4\cdot\dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} r^2 d\theta = 2\int_0^{\frac{\pi}{2}} (\cos^2\theta)^2 d\theta = 2\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = 2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{8}\pi$ 〃

(3) 曲線 C の長さを求めよ。

(解) $\dfrac{dr}{d\theta} = 2\cos\theta\cdot(\cos\theta)' = 2\cos\theta(-\sin\theta) = -2\sin\theta\cos\theta$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(\cos^2\theta)^2 + (-2\sin\theta\cos\theta)^2} = \sqrt{\cos^4\theta + 4\sin^2\theta\cos^2\theta}$$

$$= \sqrt{\cos^2\theta(\cos^2\theta + 4\sin^2\theta)x} = |\cos\theta|\sqrt{1 - \sin^2\theta + 4\sin^2\theta} = |\cos\theta|\sqrt{1 + 3\sin^2\theta}$$

$= \sqrt{3}\cdot\sqrt{\dfrac{1}{3} + \sin^2\theta}\cdot|\cos\theta|$　　ここで $0 \leqq \theta \leqq \dfrac{\pi}{2}$ では $\cos\theta \geqq 0$ より $|\cos\theta| = \cos\theta$ であるから

$$L = 4\cdot\int_0^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4\cdot\sqrt{3}\int_0^{\frac{\pi}{2}} \sqrt{\frac{1}{3} + \sin^2\theta}\cdot\cos\theta\, d\theta$$

$t = \sin\theta$　とおくと　$\dfrac{dt}{d\theta} = \cos\theta$　$\therefore dt = \cos\theta\, d\theta$

θ	$0 \to \frac{\pi}{2}$
t	$0 \to 1$

$$L = 4\sqrt{3}\int_0^1 \sqrt{\frac{1}{3} + t^2}\, dt = 4\sqrt{3}\cdot\frac{1}{2}\left[t\sqrt{\frac{1}{3} + t^2} + \frac{1}{3}\log\left|t + \sqrt{\frac{1}{3} + t^2}\right|\right]_0^1$$

$$= 2\sqrt{3}\left\{\left(\sqrt{\frac{4}{3}} + \frac{1}{3}\log\left|1 + \sqrt{\frac{4}{3}}\right|\right) - \left(\frac{1}{3}\log\left|\sqrt{\frac{1}{3}}\right|\right)\right\} = 2\sqrt{3}\left\{\frac{2}{\sqrt{3}} + \frac{1}{3}\log\left(1 + \frac{2}{\sqrt{3}}\right) - \frac{1}{3}\log\frac{1}{\sqrt{3}}\right\}$$

$$= 4 + \frac{2}{\sqrt{3}}\left(\log\frac{\sqrt{3}+2}{\sqrt{3}} - \log\frac{1}{\sqrt{3}}\right) = 4 + \frac{2}{\sqrt{3}}\log\frac{\frac{2+\sqrt{3}}{\sqrt{3}}}{\frac{1}{\sqrt{3}}} = 4 + \frac{2}{\sqrt{3}}\log(2 + \sqrt{3})$$ 〃