積分の計算《基本演習》　(NO.1)　解答例 1 枚目

1. 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{x+1}{x^2}dx$

(解)$\displaystyle\int \frac{x+1}{x^2}dx=\int\left(\frac{1}{x}+\frac{1}{x^2}\right)dx$

$\displaystyle=\int\left(\frac{1}{x}+x^{-2}\right)dx=\log|x|+\frac{1}{-2+1}x^{-2+1}+c$

$\displaystyle=\log|x|-x^{-1}+c=\log|x|-\frac{1}{x}+c$　〃

(2) $\displaystyle\int e^{3-2x}dx$

(解) $3-2x=t$ とおくと、$\displaystyle\frac{dt}{dx}=-2$ $\displaystyle\therefore -\frac{1}{2}dt=dx$

$\displaystyle\int e^{3-2x}dx=-\frac{1}{2}\int e^t dt=-\tfrac{1}{2}e^t+c$

$\displaystyle=-\frac{1}{2}e^{3-2x}+c$ 〃

(別解)$\displaystyle\int e^x dx=e^x$　だから

$\displaystyle\int e^{3-2x}dx=\frac{1}{-2}e^{3-2x}+c=-\frac{1}{2}e^{3-2x}+c$　〃

(3) $\displaystyle\int (3x+2)^4dx$

(解)$3x+2=u$ とおくと、$\displaystyle\frac{du}{dx}=3$ $\displaystyle\therefore\frac{1}{3}du=dx$

$\displaystyle\int(3x+2)^4dx=\int u^4\cdot\frac{1}{3}du=\frac{1}{3}\int u^4du$

$\displaystyle=\frac{1}{3}\cdot\frac{1}{5}u^5+c=\frac{1}{15}(3x+2)^5+c$　〃

(別解)$\displaystyle\int x^4dx=\frac{1}{4+1}x^{4+1}=\frac{1}{5}x^5$　だから

$\displaystyle\int(3x+2)^4dx$

$\displaystyle=\frac{1}{3}\cdot\frac{1}{5}(3x+2)^5+c=\frac{1}{15}(3x+2)^5+c$ 〃

(4) $\displaystyle\int \tan x dx$

(解) $\displaystyle\int\tan xdx=\int\frac{\sin x}{\cos x}=\int\left(-\frac{-\sin x}{\cos x}\right)dx$

$\displaystyle=-\int\frac{(\cos x)'}{\cos x}dx=-\log|\cos x|+c$　〃

(別解) $\cos x=t$ とおくと、

$\displaystyle\frac{dt}{dx}=-\sin x$　　$\therefore\sin xdx=-dt$

$\displaystyle\int\tan xdx=\int\frac{\sin x}{\cos x}dx=\int\frac{1}{\cos x}\cdot\sin xdx$

$\displaystyle=\int\frac{1}{t}(-1)dt=-\int\frac{1}{t}dt$

$=-\log|t|+c=-\log|\cos x|+c$ 〃

(5) $\displaystyle\int x\cos xdx$

(解)$\displaystyle\int\cos xdx=\sin x$ より、部分積分法で解くと

$\displaystyle\int x\cos xdx=x\sin x-\int(x)'\sin xdx$

$\displaystyle=x\sin x-\int\sin xdx=x\sin x+\cos x+c$　〃

(6) $\displaystyle\int\frac{x}{\sqrt{1+x}}dx$

(解) $1+x=t$ とおくと、$\displaystyle\frac{dx}{dt}=1\therefore dx=dt$

$\displaystyle\int\frac{x}{\sqrt{1+x}}dx=\int\frac{t-1}{\sqrt{t}}dt=\int\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)dt$

$\displaystyle=\int(t^{\frac{1}{2}}-t^{-\frac{1}{2}})dt=\frac{1}{\frac{3}{2}}t^{\frac{3}{2}}-\frac{1}{\frac{1}{2}}t^{\frac{1}{2}}+c=\frac{2}{3}t\sqrt{t}-2\sqrt{t}+c$

$\displaystyle=\frac{2}{3}\sqrt{t}(t-3)+c=\frac{2}{3}(x-2)\sqrt{1+x}+c$　〃

(7) $\displaystyle\int\frac{x}{\sqrt{1+x^2}}dx$

(解) $1+x^2=t$ とおくと、$\displaystyle\frac{dt}{dx}=2x$ $\displaystyle\therefore\frac{1}{2}dt=xdx$

$\displaystyle\int\frac{x}{\sqrt{1+x^2}}dx=\int\frac{1}{\sqrt{t}}\cdot\frac{1}{2}dt=\frac{1}{2}\int t^{-\frac{1}{2}}dt$

$\displaystyle=\frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}t^{-\frac{1}{2}+1}+C=\frac{1}{2}\cdot\frac{1}{\frac{1}{2}}t^{\frac{1}{2}}=\sqrt{t}+C$

$=\sqrt{1+x^2}+C$　〃

(8) $\displaystyle\int x^2e^{4x}dx$

(解)$\displaystyle\int e^{4x}dx=\frac{1}{4}e^{4x}$ より、部分積分法で解くと

$\displaystyle\int x^2e^{4x}dx=x^2\left(\frac{1}{4}e^{4x}\right)-\int(x^2)'\cdot\left(\frac{1}{4}e^{4x}\right)dx$

$\displaystyle=\frac{1}{4}x^2e^{4x}-\int 2x\cdot\frac{1}{4}e^{4x}dx=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\int xe^{4x}dx$

$\displaystyle=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\left\{x\cdot\left(\frac{1}{4}e^{4x}\right)-\int x'\left(\frac{1}{4}e^{4x}\right)dx\right\}$

$\displaystyle=\frac{1}{4}x^2e^{4x}-\frac{1}{8}xe^{4x}+\frac{1}{8}\int e^{4x}dx$

$\displaystyle=\frac{1}{4}x^2e^{4x}-\frac{1}{8}xe^{4x}+\frac{1}{32}e^{4x}+C$

$\displaystyle=\frac{1}{32}e^{4x}(8x^2-4x+1)+C$　〃

(9) $\displaystyle\int\cos^3xdx$

(解) 与式$\displaystyle=\int\cos^2x\cos xdx=\int(1-\sin^2x)\cos xdx$

$\sin x=t$　とおくと、$\displaystyle\frac{dt}{dx}=\cos x\therefore dt=\cos xdx$

与式$\displaystyle=\int(1-t^2)dt=t-\frac{1}{3}t^3+C$

$\displaystyle=\sin x-\frac{1}{3}\sin^3x+C$　〃

積分の計算《基本演習》 (NO.1) 解答例 2 枚目

2. 次の定積分の値を求めよ。

(1) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(\sin x+\cos x)dx$

(解)$\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(\sin x+\cos x)dx=\Big[-\cos x+\sin x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$=(-\cos\frac{\pi}{3}+\sin\frac{\pi}{3})-(-\cos\frac{\pi}{6}+\sin\frac{\pi}{6})$

$=(-\frac{1}{2}+\frac{\sqrt{3}}{2})-(-\frac{\sqrt{3}}{2}+\frac{1}{2})=\sqrt{3}-1$ 〃

(2) $\displaystyle\int_1^2\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$

(解) 与式 $\displaystyle=\int_1^2\left(x-2+\frac{1}{x}\right)dx=\left[\frac{1}{2}x^2-2x+\log|x|\right]_1^2$

$=(2-4+\log 2)-(\frac{1}{2}-2+\log 1)=\log 2-\frac{1}{2}$ 〃

(3) $\displaystyle\int_0^4\sqrt{2x+1}dx$

x	$0\ \rightarrow\ 4$
t	$1\ \rightarrow\ 9$

(解) $2x+1=t$ とおくと、

$\dfrac{dt}{dx}=2\quad\therefore dx=\dfrac{1}{2}dt$

$\displaystyle\int_0^4\sqrt{2x+1}dx=\int_1^9\sqrt{t}\cdot\frac{1}{2}dt=\frac{1}{2}\int_1^9 t^{\frac{1}{2}}dt$

$\displaystyle=\frac{1}{2}\left[\frac{1}{\frac{3}{2}}t^{\frac{3}{2}}\right]_1^9=\frac{1}{3}\Big[t\sqrt{t}\Big]_1^9=\frac{1}{3}(9\sqrt{9}-1\sqrt{1})=\frac{26}{3}$ 〃

(4) $\displaystyle\int_0^1 xe^{-x}dx$

(解) $\displaystyle\int_0^1 e^{-x}dx=\frac{1}{-1}e^{-x}=-e^{-x}$ だから部分積分法で

$\displaystyle\int_0^1 xe^{-x}dx=\Big[x(-e^{-x})\Big]_0^1-\int_0^1(x)'(-e^{-x})dx$

$\displaystyle=\Big[-xe^{-x}\Big]_0^1+\int_0^1 e^{-x}dx$

$=\Big[-xe^{-x}\Big]_0^1+\Big[-e^{-x}\Big]_0^1$

$=\{(-1\cdot e^{-1})-(-0\cdot e^0)\}+\{(-e^{-1})-(-e^0)\}$

$=-\dfrac{1}{e}-\dfrac{1}{e}+1=1-\dfrac{2}{e}$ 〃

(5) $\displaystyle\int_0^{\frac{\pi}{6}}\sin 2x dx$

(解) $2x=t$ とおくと

x	$0\ \rightarrow\ \frac{\pi}{6}$
t	$0\ \rightarrow\ \frac{\pi}{3}$

$\dfrac{dt}{dx}=2\qquad\therefore\dfrac{1}{2}dt-dx$

$\displaystyle\int_0^{\frac{\pi}{6}}\sin 2xdx=\int_0^{\frac{\pi}{3}}\sin t\cdot\frac{1}{2}dt=\frac{1}{2}\int_0^{\frac{\pi}{3}}\sin tdt$

$=\dfrac{1}{2}\Big[-\cos t\Big]_0^{\frac{\pi}{3}}=-\dfrac{1}{2}\Big[\cos t\Big]_0^{\frac{\pi}{3}}$

$=-\dfrac{1}{2}\left(\cos\dfrac{\pi}{3}-\cos 0\right)=-\dfrac{1}{2}\left(\dfrac{1}{2}-1\right)=\dfrac{1}{4}$ 〃

(別解)与式 $=\left[\dfrac{1}{2}\cdot(-\cos 2x)\right]_0^{\frac{\pi}{6}}=-\dfrac{1}{2}\Big[\cos 2x\Big]_0^{\frac{\pi}{6}}$

$=-\dfrac{1}{2}\left(\cos\dfrac{\pi}{3}-\cos 0\right)=-\dfrac{1}{2}\left(\dfrac{1}{2}-1\right)=\dfrac{1}{4}$ 〃

(6) $\displaystyle\int_1^3\log xdx$

(解)$\displaystyle\int 1dx=x$ だから部分積分法で解くと

$\displaystyle\int_1^3\log xdx=\int_1^3 1\cdot\log xdx$

$\displaystyle=\Big[x\log x\Big]_1^3-\int_1^3 x(\log x)'dx$

$\displaystyle=\Big[x\log x\Big]_1^3-\int_1^3 x\cdot\frac{1}{x}dx$

$\displaystyle=\Big[x\log x\Big]_1^3-\int_1^3 1dx=\Big[x\log x\Big]_1^3-\Big[x\Big]_1^3$

$=(3\log 3-1\log 1)-(3-1)=3\log 3-2$ 〃

(7) $\displaystyle\int_{-1}^1\sqrt{4-x^2}dx$ 　$\left(f(x)=\sqrt{4-x^2}\ \text{は偶関数である。}\right)$

(解)$\displaystyle\int_{-1}^1\sqrt{4-x^2}dx=2\int_0^1\sqrt{2^2-x^2}dx$

$=2\left[\dfrac{1}{2}\left\{x\sqrt{2^2-x^2}+2^2\mathrm{Sin}^{-1}\dfrac{x}{2}\right\}\right]_0^1$

$=\left[x\sqrt{2^2-x^2}+2^2\mathrm{Sin}^{-1}\dfrac{x}{2}\right]_0^1$

$=\left(\sqrt{3}+4\mathrm{Sin}^{-1}\dfrac{1}{2}\right)-(0-4\mathrm{Sin}^{-1}0)$

$=\sqrt{3}+4\cdot\dfrac{\pi}{6}+4\cdot 0=\sqrt{3}+\dfrac{2}{3}\pi$ 〃

(別解)

与式 $\displaystyle=2\int_0^1\sqrt{2^2-x^2}dx$

$x=2\sin\theta\left(0\leqq\theta\leqq\dfrac{\pi}{2}\right)$ とおくと

$\dfrac{dx}{d\theta}=2\cos\theta\quad\therefore dx=2\cos\theta d\theta$

x	0	$\rightarrow$	1
θ	0	$\rightarrow$	$\frac{\pi}{6}$

$\sqrt{2^2-x^2}=\sqrt{4-4\sin^2\theta}=2\sqrt{1-\sin^2\theta}$

$=2|\cos\theta|=2\cos\theta$

与式 $\displaystyle=2\int_0^{\frac{\pi}{6}}2\cos\theta\cdot 2\cos\theta d\theta$

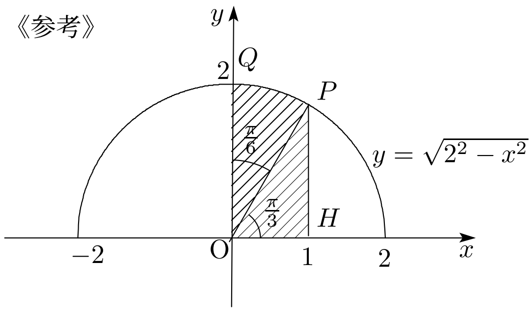
$\displaystyle=8\int_0^{\frac{\pi}{6}}\cos^2\theta d\theta=8\int_0^{\frac{\pi}{6}}\frac{1+\cos 2\theta}{2}d\theta$

$\displaystyle=4\int_0^{\frac{\pi}{6}}(1+\cos 2\theta)d\theta=4\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$

$=4\left\{\left(\dfrac{\pi}{6}+\dfrac{1}{2}\sin\dfrac{\pi}{3}-(0-0)\right)\right\}$

$=4\left(\dfrac{\pi}{6}+\dfrac{1}{2}\cdot\dfrac{\sqrt{3}}{2}\right)=\dfrac{2}{3}\pi+\sqrt{3}$ 〃

《参考》



斜線の部分の面積を S とすると

$S=\triangle OPH+$ 扇形 OPQ

$=\frac{1}{2}\cdot 1\cdot 2\sin\frac{\pi}{3}+\frac{1}{2}\cdot 2^2\cdot\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{\pi}{3}$

$\int_{-1}^{1}\sqrt{4-x^2}dx=2\int_0^1\sqrt{2^2-x^2}dx$

$=2S=2\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)=\sqrt{3}+\frac{2}{3}\pi$

3. 極限値 $\lim_{n\to\infty}\frac{1}{n^7}\sum_{k=1}^{n}k^6$ の値を求めよ。

(解) 区間 $[0,1]$ を n 等分して n 個の小区分に分けて

$x_k=\frac{1-0}{n}\cdot k=\frac{k}{n},\quad \Delta x_k=\frac{1-0}{n}=\frac{1}{n}$

$(n=1,2,3,\cdots,n)$ とおくと、

定積分の定義によって

$\lim_{n\to\infty}\frac{1}{n^7}\sum_{k=1}^{n}k^6=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^6\cdot\frac{1}{n}$

$=\lim_{\Delta x_k\to 0}\sum_{k=1}^{n}x_k^{\,6}\Delta x_k=\int_0^1 x^6dx$

$=\left[\frac{1}{7}x^7\right]_0^1=\frac{1}{7}\left[x^7\right]_0^1=\frac{1}{7}(1^7-0^7)=\frac{1}{7}$ 〃

4. $\int_1^x f(t)dt=x^2-ax+2$ を満たす関数 $f(x)$ と定数 a を求めよ。

(解)$\frac{d}{dx}\int_1^x f(t)dt=\frac{d}{dx}(x^2-ax+2)$

$f(x)=2x-a\qquad\therefore f(t)=2t-a$

$\int_1^x f(t)dt=\int_1^x(2t-a)dt=\left[t^2-at\right]_1^x$

$=(x^2-ax)-(1^2-a\cdot 1)=x^2-ax+a-1$

これと与式を比べて　　$a-1=2$

$\therefore a=3$ 〃 ,　　$\therefore f(x)=2x-3$ 〃