第2章　微分積分 II《§3　重積分》

101(3) 領域 $D=\left\{(x,y)\middle|x+y\geqq 0,\ x^2+y^2\leqq 1\right\}$ のとき,

重積分 $\displaystyle\iint_D \log(1+x^2+y^2)dxdy$ の値を求めよ.

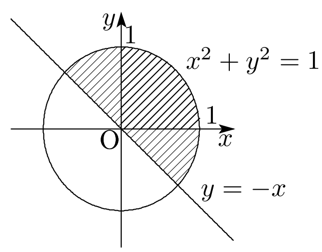
(神戸大)

《　ポイント：$\left(\dfrac{1+r^2}{2}\right)'=r$　》

(解)

$x=r\cos\theta,\ y=r\sin\theta(r\geqq 0)$ とおくと,

領域 D は図の斜線の部分であるから,



$0\leqq r\leqq 1,\ -\dfrac{\pi}{4}\leqq\theta\leqq\dfrac{3}{4}\pi$

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{vmatrix}=r$$

$$\therefore\ \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$$

$$\iint_D \log(1+x^2+y^2)dxdy$$

$$=\iint_D \log(1+r^2)rdrd\theta$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\left\{\int_0^1 r\log(1+r^2)dr\right\}d\theta$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}d\theta\int_0^1 r\log(1+r^2)dr$$

$$=\Big[\ \theta\ \Big]_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\cdot\int_0^1\left(\frac{1+r^2}{2}\right)'\log(1+r^2)dr$$

$$=\pi\left\{\left[\frac{1+r^2}{2}\cdot\log(1+r^2)\right]_0^1-\int_0^1\frac{1+r^2}{2}\cdot\big(\log(1+r^2)\big)'dr\right\}$$

$$=\pi\left\{\left(\frac{2}{2}\log 2-\frac{1}{2}\log 1\right)-\int_0^1\frac{1+r^2}{2}\cdot\frac{1}{1+r^2}\cdot 2rdr\right\}$$

$$=\pi\left\{\log 2-\int_0^1 rdr\right\}=\pi\left\{\log 2-\left[\frac{r^2}{2}\right]_0^1\right\}$$

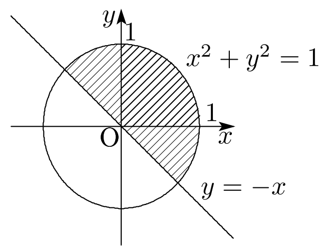
$$=\pi\left(\log 2-\frac{1}{2}\right)\quad〃$$

《　ポイント：$t = 1 + r^2$ として置換積分をしてもよい．　》

(別解)

$x = r\cos\theta,\ y = r\sin\theta (r \geqq 0)$　とおくと，

領域 D は図の斜線の部分であるから，



$0 \leqq r \leqq 1,\ -\dfrac{\pi}{4} \leqq \theta \leqq \dfrac{3}{4}\pi$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore\ \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = r$$

$$I = \iint_D \log(1 + x^2 + y^2)dxdy$$

$$= \iint_D \log(1 + r^2)rdrd\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \left\{\int_0^1 \log(1 + r^2)rdr\right\} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_0^1 \log(1 + r^2)rdr$$

$$= \Big[\ \theta\ \Big]_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \cdot \int_0^1 \log(1 + r^2)rdr$$

$$= \pi \int_0^1 \log(1 + r^2)rdr$$

$t = 1 + r^2$ とおくと，　$\dfrac{dt}{dr} = 2r$　$\therefore\ \dfrac{1}{2}dt = rdr$

r	$0 \to 1$
t	$1 \to 2$

$$I = \pi \int_1^2 \log t \cdot \frac{1}{2}dt = \frac{\pi}{2}\int_1^2 (t)' \cdot \log t dt$$

$$= \frac{\pi}{2}\left\{\Big[\ t \cdot \log t\ \Big]_1^2 - \int_1^2 t \cdot (\log t)' dt\right\}$$

$$= \frac{\pi}{2}\left\{(2 \cdot \log 2 - 1 \cdot \log 1) - \int_1^2 t \cdot \frac{1}{t}dt\right\}$$

$$= \frac{\pi}{2}\left\{2\log 2 - \int_1^2 dt\right\} = \frac{\pi}{2}\left\{2\log 2 - \Big[\ t\ \Big]_1^2\right\}$$

$$= \frac{\pi}{2}(2\log 2 - 1) = \pi\left(\log 2 - \frac{1}{2}\right)$$　〃