第４章　応用数学《§2　複素関数》

246 複素関数 $f(z) = \dfrac{8}{2z^4 + 1 - \sqrt{3}i}$ について，以下の問に答えよ．

(1) $f(z)$ の特異点をすべて求めよ．

(2) $f(z)$ の各特異点 α に対して，その留数 $\mathrm{Res}[\alpha]$ を求めよ．

(3) 複素積分 $\displaystyle\int_{|z-1|=\frac{2}{3}} f(z)dz$ を求めよ．ただし，積分路は正の向きに一周するものとする．

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《　ポイント：留数定理　》

単純閉曲線 C の内部にある特異点 $\alpha_1,\ \alpha_2,\ \cdots\ \alpha_n$ を除き，C の周および内部で $f(z)$ が正則ならば，

$\displaystyle\int_C f(z)dz = 2\pi i\Big(\mathrm{Res}[f,\ \alpha_1] + \mathrm{Res}[f,\ \alpha_2] + \cdots + \mathrm{Res}[f,\ \alpha_n]\Big)$

(解)

(1) 《　ポイント：分母が 0 になる z を求める．　　また，$e^{i\theta} = \cos\theta + i\sin\theta$ に留意する，　》

$2z^4 + 1 - \sqrt{3}i = 0$ とおくと，$z^4 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

$z = re^{i\theta}$ とおくと，$z^4 = r^4\left\{e^{i\theta}\right\}^4 = r^4e^{4\theta i}$

$-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i = \cos\left(\dfrac{2}{3}\pi + 2n\pi\right) + i\sin\left(\dfrac{2}{3}\pi + 2n\pi\right) = e^{\left(\frac{2}{3}+2n\pi\right)i}$　(n は整数)

$\therefore r^4e^{4\theta i} = e^{\left(\frac{2}{3}+2n\pi\right)i}$

よって，$r = 1,\ \theta = \dfrac{\pi}{6} + \dfrac{n\pi}{2}$　$(n = 0, 1, 2, 3)$ である．

$n = 0$ のとき，$\theta = \dfrac{\pi}{6}$　　$n = 1$ のとき，$\theta = \dfrac{\pi}{6} + \dfrac{\pi}{2} = \dfrac{2\pi}{3}$

$n = 2$ のとき，$\theta = \dfrac{\pi}{6} + \pi = \dfrac{7\pi}{6}$　　$n = 3$ のとき，$\theta = \dfrac{\pi}{6} + \dfrac{3\pi}{2} = \dfrac{5\pi}{3}$

これから，$z = e^{\frac{\pi}{6}i},\ e^{\frac{2\pi}{3}i},\ e^{\frac{7\pi}{6}i},\ e^{\frac{5\pi}{3}i}$ であるから，

$z = e^{\frac{\pi}{6}i} = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

$z = e^{\frac{2\pi}{3}i} = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3} = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

$z = e^{\frac{7\pi}{6}i} = \cos\dfrac{7\pi}{6} + i\sin\dfrac{7\pi}{6} = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

$z = e^{\frac{5\pi}{3}i} = \cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

よって，特異点は，$\dfrac{\sqrt{3}+i}{2}$　$\dfrac{-1+\sqrt{3}i}{2}$，$\dfrac{-\sqrt{3}-i}{2}$，$\dfrac{1-\sqrt{3}i}{2}$　〃

(解)

(2) $f(z)=\dfrac{4}{z^4+\dfrac{1-\sqrt{3}i}{2}}=\dfrac{4}{\left(z-\dfrac{\sqrt{3}+i}{2}\right)\left(z-\dfrac{-1+\sqrt{3}i}{2}\right)\left(z-\dfrac{-\sqrt{3}-i}{2}\right)\left(z-\dfrac{1-\sqrt{3}i}{2}\right)}$

$$\mathrm{Res}\left[\frac{\sqrt{3}+i}{2}\right]=\lim_{z\to\frac{\sqrt{3}+i}{2}}\left(z-\frac{\sqrt{3}+i}{2}\right)f(z)$$

$$=\frac{4}{\left(\dfrac{\sqrt{3}+i}{2}-\dfrac{-1+\sqrt{3}i}{2}\right)\left(\dfrac{\sqrt{3}+i}{2}-\dfrac{-\sqrt{3}-i}{2}\right)\left(\dfrac{\sqrt{3}+i}{2}-\dfrac{1-\sqrt{3}i}{2}\right)}$$

$$=\frac{16}{\{(\sqrt{3}+1)-(\sqrt{3}-1)i\}\{(\sqrt{3}-1)+(\sqrt{3}+1)i\}(\sqrt{3}+i)}$$

$$=\frac{16}{(4+4\sqrt{3}i)(\sqrt{3}+i)}=\frac{4}{(1+\sqrt{3}i)(\sqrt{3}+i)}=\frac{4}{4i}=\frac{i}{i^2}=-i \quad〃$$

$$\mathrm{Res}\left[\frac{-1+\sqrt{3}i}{2}\right]=\lim_{z\to\frac{-1+\sqrt{3}i}{2}}\left(z-\frac{-1+\sqrt{3}i}{2}\right)f(z)$$

$$=\frac{4}{\left(\dfrac{-1+\sqrt{3}i}{2}-\dfrac{\sqrt{3}+i}{2}\right)\left(\dfrac{-1+\sqrt{3}i}{2}-\dfrac{-\sqrt{3}-i}{2}\right)\left(\dfrac{-1+\sqrt{3}i}{2}-\dfrac{1-\sqrt{3}i}{2}\right)}$$

$$=\frac{16}{\{-(\sqrt{3}+1)+(\sqrt{3}-1)i\}\{(\sqrt{3}-1)+(\sqrt{3}+1)i\}(-1+\sqrt{3}i)}$$

$$=\frac{16}{(-4-4\sqrt{3}i)(-1+\sqrt{3}i)}=\frac{4}{(1+\sqrt{3}i)(1-\sqrt{3}i)}=\frac{4}{1-3i^2}=1 \quad〃$$

$$\mathrm{Res}\left[\frac{-\sqrt{3}-i}{2}\right]=\lim_{z\to\frac{-\sqrt{3}-i}{2}}\left(z-\frac{-\sqrt{3}-i}{2}\right)f(z)$$

$$=\frac{4}{\left(\dfrac{-\sqrt{3}-i}{2}-\dfrac{\sqrt{3}+i}{2}\right)\left(\dfrac{-\sqrt{3}-i}{2}-\dfrac{-1+\sqrt{3}i}{2}\right)\left(\dfrac{-\sqrt{3}-i}{2}-\dfrac{1-\sqrt{3}i}{2}\right)}$$

$$=\frac{16}{(-\sqrt{3}-i)\{-(\sqrt{3}-1)-(\sqrt{3}+1)i\}\{-(\sqrt{3}+1)+(\sqrt{3}-1)i\}}$$

$$=\frac{16}{-(\sqrt{3}+i)(4+4\sqrt{3}i)}=\frac{4}{-(\sqrt{3}+i)(1+\sqrt{3}i)}=-\frac{4}{4i}=-\frac{i}{i^2}=i \quad〃$$

$$\mathrm{Res}\left[\frac{1-\sqrt{3}i}{2}\right]=\lim_{z\to\frac{1-\sqrt{3}i}{2}}\left(z-\frac{1-\sqrt{3}i}{2}\right)f(z)$$

$$=\frac{4}{\left(\dfrac{1-\sqrt{3}i}{2}-\dfrac{\sqrt{3}+i}{2}\right)\left(\dfrac{1-\sqrt{3}i}{2}-\dfrac{-1+\sqrt{3}i}{2}\right)\left(\dfrac{1-\sqrt{3}i}{2}-\dfrac{-\sqrt{3}-i}{2}\right)}$$

$$=\frac{16}{\{-(\sqrt{3}-1)-(\sqrt{3}+1)i\}\{(\sqrt{3}+1)-(\sqrt{3}+1)i\}(1-\sqrt{3}i)}$$

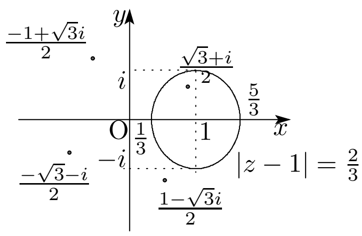
$$=\frac{16}{(-4-4\sqrt{3}i)(1-\sqrt{3}i)}=\frac{4}{-(1+\sqrt{3}i)(1-\sqrt{3}i)}=\frac{4}{-(1-3i^2)}=-1 \quad〃$$

(3) $f(z)$ の特異点のうち，円 $|z-1| = \dfrac{2}{3}$ の内部にあるものは

$\dfrac{\sqrt{3}+i}{2}$ だけである．

よって，留数定理より，

$$\int_{|z-1|=\frac{2}{3}} f(z)dz = 2\pi i \left(\mathrm{Res}\left[\frac{\sqrt{3}+i}{2}\right] \right) = 2\pi i \cdot (-i) = 2\pi \quad 〃$$



(別解)

《　ポイント：ロピタルの定理を利用する．　また，(1) の解答を利用する．　》

(2) $\mathrm{Res}\left[\dfrac{\sqrt{3}+i}{2}\right] = \mathrm{Res}\left[e^{\frac{\pi}{6}i}\right] = \displaystyle\lim_{z\to e^{\frac{\pi}{6}i}} \left(z - e^{\frac{\pi}{6}i}\right) f(z)$

$$= \lim_{z\to e^{\frac{\pi}{6}i}} \frac{8\left(z-e^{\frac{\pi}{6}i}\right)}{2z^4+1+\sqrt{3}i} = \lim_{z\to e^{\frac{\pi}{6}i}} \frac{8}{8z^3} = \lim_{z\to e^{\frac{\pi}{6}i}} z^{-3} = \left(e^{\frac{\pi}{6}i}\right)^{-3} = e^{-\frac{\pi}{2}i}$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i\cdot(-1) = -i \quad 〃$$

$\mathrm{Res}\left[\dfrac{-1+\sqrt{3}i}{2}\right] = \mathrm{Res}\left[e^{\frac{2\pi}{3}i}\right] = \displaystyle\lim_{z\to e^{\frac{2\pi}{3}i}} \left(z - e^{\frac{2\pi}{3}i}\right) f(z)$

$$= \lim_{z\to e^{\frac{2\pi}{3}i}} \frac{8\left(z-e^{\frac{2\pi}{3}i}\right)}{2z^4+1+\sqrt{3}i} = \lim_{z\to e^{\frac{2\pi}{3}i}} \frac{8}{8z^3} = \lim_{z\to e^{\frac{2\pi}{3}i}} z^{-3} = \left(e^{\frac{2\pi}{3}i}\right)^{-3} = e^{-2\pi i}$$

$$= \cos(-2\pi) + i\sin(-2\pi) = 1 + i\cdot 0 = 1 \quad 〃$$

$\mathrm{Res}\left[\dfrac{-\sqrt{3}-i}{2}\right] = \mathrm{Res}\left[e^{\frac{7\pi}{6}i}\right] = \displaystyle\lim_{z\to e^{\frac{7\pi}{6}i}} \left(z - e^{\frac{7\pi}{6}i}\right) f(z)$

$$= \lim_{z\to e^{\frac{7\pi}{6}i}} \frac{8\left(z-e^{\frac{7\pi}{6}i}\right)}{2z^4+1+\sqrt{3}i} = \lim_{z\to e^{\frac{7\pi}{6}i}} \frac{8}{8z^3} = \lim_{z\to e^{\frac{7\pi}{6}i}} z^{-3} = \left(e^{\frac{7\pi}{6}i}\right)^{-3} = e^{-\frac{7\pi}{2}}$$

$$= \cos\left(-\frac{7\pi}{2}\right) + i\sin\left(-\frac{7\pi}{2}\right) = \cos\left(\frac{\pi}{2}-4\pi\right) + i\sin\left(\frac{\pi}{2}-4pi\right) = 0 + i\cdot 1 = i \quad 〃$$

$\mathrm{Res}\left[\dfrac{1-\sqrt{3}i}{2}\right] = \mathrm{Res}\left[e^{\frac{5\pi}{3}i}\right] = \displaystyle\lim_{z\to e^{\frac{5\pi}{3}i}} \left(z - e^{\frac{5\pi}{3}i}\right) f(z)$

$$= \lim_{z\to e^{\frac{5\pi}{3}i}} \frac{8\left(z-e^{\frac{5\pi}{3}i}\right)}{2z^4+1+\sqrt{3}i} = \lim_{z\to e^{\frac{5\pi}{3}i}} \frac{8}{8z^3} = \lim_{z\to e^{\frac{5\pi}{3}i}} z^{-3} = \left(e^{\frac{5\pi}{3}i}\right)^{-3} = e^{-5\pi i}$$

$$= \cos(-5\pi) + i\sin(-5\pi) = \cos(\pi-6\pi) + i\sin(\pi-6pi) = -1 + i\cdot 0 = -1 \quad 〃$$