第２章　微分積分 II《§3　重積分》

116 $D=\{(x,y,z)\in\mathbb{R}^3 \mid x+y\geqq 0\}$ と定めるとき,

積分 $\displaystyle\iiint_D \frac{dxdydz}{(x^2+y^2+z^2+1)^2}$ を求めよ.

(筑波大)

《　ポイント：３次元極座標　》

$x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\theta$

$0\leqq\theta\leqq\pi,\ 0\leqq\varphi\leqq 2\pi$

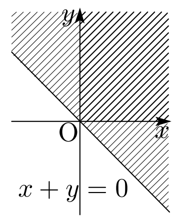
ヤコビアン は,

$$J(r,\theta,\varphi)=\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=\begin{vmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta\\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0\end{vmatrix}=r^2\sin\theta\geqq 0$$

(解)

$x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\theta$ とおくと,

領域 D は図の斜線の部分であるから,



$0\leqq r<\infty,\ 0\leqq\theta\leqq\pi,\ -\frac{\pi}{4}\leqq\varphi\leqq\frac{3}{4}\pi$

ヤコビアンは $J(r,\theta,\varphi)=r^2\sin\theta$

$x^2+y^2+z^2=r^2\sin^2\theta\cos^2\varphi+r^2\sin^2\theta\sin^2\varphi+r^2\cos^2\theta$

$=r^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi)+r^2\cos^2\theta$

$=r^2\sin^2\theta+r^2\cos^2\theta=r^2(\sin^2\theta+\cos^2\theta)=r^2$

$$I=\iiint_D\frac{dxdydz}{(x^2+y^2+z^2+1)^2}$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\left\{\int_0^\pi\left(\int_0^\infty\frac{1}{(r^2+1)^2}\cdot r^2\sin\theta dr\right)d\theta\right\}d\varphi$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\left\{\int_0^\pi\left(\int_0^\infty\frac{1}{(r^2+1)^2}\cdot r^2 dr\right)\sin\theta\,d\theta\right\}d\varphi$$

ここで, $r=\tan v$ とおくと,

$\dfrac{dr}{dv}=\dfrac{1}{\cos^2 v}\quad\therefore dr=\dfrac{1}{\cos^2 v}dv\qquad r^2+1=\tan^2 v+1=\dfrac{1}{\cos^2 v}$

r	$0\to\infty$
v	$0\to\frac{\pi}{2}$

$$\int_0^\infty\frac{1}{(r^2+1)^2}r^2\,dr=\int_0^{\frac{\pi}{2}}\cos^4 v\cdot\tan^2 v\cdot\frac{1}{\cos^2 v}\,dv=\int_0^{\frac{\pi}{2}}\sin^2 v\,dv=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

$$\therefore I=\frac{\pi}{4}\cdot\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\left\{\int_0^\pi\sin\theta\,d\theta\right\}d\varphi$$

ここで, $\displaystyle\int_0^\pi\sin\theta\,d\theta=-\Big[\cos\theta\Big]_0^\pi=-(\cos\pi-\cos 0)=-(-1-1)=2$ であるから,

$$\therefore I=\frac{\pi}{4}\cdot 2\cdot\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}d\varphi=\frac{\pi}{2}\cdot\Big[\,\varphi\,\Big]_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}=\frac{\pi}{2}\left\{\frac{3}{4}\pi-\left(-\frac{\pi}{4}\right)\right\}=\frac{\pi}{2}\cdot\pi=\frac{\pi^2}{2}\quad〃$$