第３章　線形代数《§1　ベクトル》

$\boxed{156}$ ３次元数ベクトル空間を $\mathbb{R}^3$ とする，次の $\mathbb{R}^3$ のベクトル $\{\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3\}$ から，グラム・シュミットの正規直交化法によって，$\mathbb{R}^3$ の正規直交系 $\{\boldsymbol{u}_1,\ \boldsymbol{u}_2,\ \boldsymbol{u}_3\}$ を構成しなさい.

$$\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{a}_3 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix},$$

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(解)

$$\boldsymbol{u}_1 = \frac{1}{|\boldsymbol{a}_1|}\boldsymbol{a}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad /\!/$$

$$\boldsymbol{b}_2 = \boldsymbol{a}_2 - (\boldsymbol{a}_2 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\sqrt{3} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \boldsymbol{u}_2 = \frac{1}{|\boldsymbol{b}_2|}\boldsymbol{b}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad /\!/$$

$$\boldsymbol{b}_3 = \boldsymbol{a}_3 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_2)\boldsymbol{u}_2$$

$$= \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} - \left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left(2\sqrt{2}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$= \frac{1}{3}\left\{\begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix}\right\} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

ここで,

$$|\boldsymbol{b}_3| = \frac{1}{3} \cdot \sqrt{6} = \frac{\sqrt{6}}{3}$$

であるから,

$$\therefore \boldsymbol{u}_3 = \frac{1}{|\boldsymbol{b}_3|}\boldsymbol{b}_3 = \frac{3}{\sqrt{6}} \cdot \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad /\!/$$

《答の確認》

$$|\boldsymbol{a}_1| = \left| \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| = \frac{1}{\sqrt{3}} \cdot \sqrt{3} = 1$$

$$|\boldsymbol{a}_2| = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right| = \frac{1}{\sqrt{2}} \cdot \sqrt{2} = 1$$

$$|\boldsymbol{a}_3| = \left| \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right| = \frac{1}{\sqrt{6}} \cdot \sqrt{6} = 1$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \cdot 0 = 0 \qquad \therefore \boldsymbol{u}_1 \perp \boldsymbol{u}_2$$

$$\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \cdot 0 = 0 \qquad \therefore \boldsymbol{u}_2 \perp \boldsymbol{u}_3$$

$$\boldsymbol{u}_3 \cdot \boldsymbol{u}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3\sqrt{2}} \cdot 0 = 0 \qquad \therefore \boldsymbol{u}_3 \perp \boldsymbol{u}_1$$

《　ポイント：グラム・シュミットの正規直交化法　》

与えられた $\mathbb{R}^3$ のベクトル $\{\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3\}$ から，次のようにして正規直交系

（大きさ１のベクトルで，互いに直交するベクトルの組）$\{\boldsymbol{u}_1,\ \boldsymbol{u}_2,\ \boldsymbol{u}_3\}$ を作る方法.

[注意] $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = \boldsymbol{u}_3 \cdot \boldsymbol{u}_1 = 0 \qquad \boldsymbol{u}_1 \cdot \boldsymbol{u}_1 = \boldsymbol{u}_2 \cdot \boldsymbol{u}_2 = \boldsymbol{u}_3 \cdot \boldsymbol{u}_3 = 1$

① $\boldsymbol{u}_1 = \dfrac{\boldsymbol{a}_1}{|\boldsymbol{a}_1|}$ とする.

[確認] $\boldsymbol{u}_1$は$\boldsymbol{a}_1$の大きさを1にしたもの.

② $\boldsymbol{b}_2 = \boldsymbol{a}_2 - (\boldsymbol{a}_2 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1$ とおき，　$\boldsymbol{u}_2 = \dfrac{\boldsymbol{b}_2}{|\boldsymbol{b}_2|}$ とする.

[確認] $\boldsymbol{b}_2$ は $\boldsymbol{a}_1$ と $\boldsymbol{a}_2$ に張られる平面上にあり，$\boldsymbol{u}_1$ に垂直なもの.

$$\boldsymbol{b}_2 \cdot \boldsymbol{u}_1 = \boldsymbol{a}_2 \cdot \boldsymbol{u}_1 - (\boldsymbol{a}_2 \cdot \boldsymbol{u}_1)(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) = \boldsymbol{a}_2 \cdot \boldsymbol{u}_1 - \boldsymbol{a}_2 \cdot \boldsymbol{u}_1 = 0 \qquad \therefore \boldsymbol{b}_2 \perp \boldsymbol{u}_1$$

③ $\boldsymbol{b}_3 = \boldsymbol{a}_3 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_2)\boldsymbol{u}_2$ とおき，　$\boldsymbol{u}_3 = \dfrac{\boldsymbol{b}_3}{|\boldsymbol{b}_3|}$ とする.

[確認] $\boldsymbol{b}_3$ は $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ に垂直なもの.

$$\boldsymbol{b}_3 \cdot \boldsymbol{u}_1 = \boldsymbol{a}_3 \cdot \boldsymbol{u}_1 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_1)(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_2)(\boldsymbol{u}_2 \cdot \boldsymbol{u}_1) = \boldsymbol{a}_3 \cdot \boldsymbol{u}_1 - \boldsymbol{a}_3 \cdot \boldsymbol{u}_1 = 0 \qquad \therefore \boldsymbol{b}_3 \perp \boldsymbol{u}_1$$

$$\boldsymbol{b}_3 \cdot \boldsymbol{u}_2 = \boldsymbol{a}_3 \cdot \boldsymbol{u}_2 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_1)(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2) - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_2)(\boldsymbol{u}_2 \cdot \boldsymbol{u}_2) = \boldsymbol{a}_3 \cdot \boldsymbol{u}_2 - \boldsymbol{a}_3 \cdot \boldsymbol{u}_2 = 0 \qquad \therefore \boldsymbol{b}_3 \perp \boldsymbol{u}_2$$